MATHEMATICS OF MODERN ENGINEERING

VOLUME I

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PREFACE

The purpose of this text is to facilitate the study and use of mathematics with especial reference to its applications in engineering. The justification of another text in this general field, however, is not its purpose merely, which is broadly the same as that of any book on the same subject, but the point of view from which it is written. The latter has been tempered, first, by the joint participation of a mathematician who has worked in engineering and of an engineer who has worked with mathematics; and secondly, by the atmosphere of the engineering office and the classroom. On its educational side, the text represents a gradual development of material and form during a classroom experience of ten years; and, on the engineering side, embodies a scope and method suggested by engineering work extending over a much longer period. It is thus neither a text on mathematics nor one on engineering, nor yet merely a handbook on engineering mathematics. Rather, it is, first, a guide in bridging the gap in engineering between physics and mathematics by the scientific method; and secondly, a presentation, suitable for engineers, of those aspects of mathematics which the experience of a large manufacturing organization in dealing with electrical and mechanical problems has indicated to be of value to engineers.

Such a book may appear choppy and lacking in unity from the mathematician's point of view. However, the unity of the text lies in its method of approach in the mathematical formulation of engineering problems. The problems themselves and the mathematical devices used in solving them afford a great variety, and it is this variety that may, in a superficial review, present the appearance of choppiness.

We believe that there is a broad educational field for a text which outlines the application of mathematics from the point of view referred to, and which is of proved usefulness in class work. The text was originally developed for the Advanced Course in Engineering of the General Electric Company. Although prepared for that course, which comprises a selected group of engineering graduates just out of college, nevertheless the book should be useful as well in college in both undergraduate and graduate engineering work. This volume covers ground which those students who are interested in the higher

levels of engineering service should study as undergraduates. The material of the second volume fits more appropriately in graduate work. The only knowledge presupposed for understanding Volume I is the calculus.

On account of the special nature and purpose of the text it has seemed desirable to include a **Foreword for Instructors** which may be helpful to them in presenting the point of view to the student.

Assistance from engineers of the General Electric Company is acknowledged. They have suggested problem material from actual engineering work, and offered valuable criticisms. We are especially indebted to Dr. A. R. Stevenson and Messrs. P. L. Alger, E. E. Johnson, and Alan Howard of the General Electric Company.

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FOREWORD FOR INSTRUCTORS

The great increase in the use of mathematics in engineering during the last decade or so is merely one phase of a general increase throughout the whole field of applied mathematics. In the latter field, tensor analysis is now the everyday language of the physicist; Hamilton's principle, involving the calculus of variations, is one of the chief stand-by theorems of mathematical physics. Courant and Hilbert's *Methoden der mathematischen Physik* is quite as much a textbook of mathematics as it is of physics. The development of wave mechanics has made the concepts of groups, Hermitian matrices, Eigenwerte, extremals, invariants, and contact transformations, and many other similar concepts, common property of the chemist as well as of the physicist and mathematician. Modern theories of ballistics are concerned almost entirely with the solution of certain differential equations. Extensions have not stopped at the boundaries of physical science. The mathematical developments in political economy, statistics, and life insurance form a long list in the *Encyklopädie der mathematischen Wissenschaften*, and the applications in biology, and psychology become increasingly more numerous.

The engineering phase of this growth is far-reaching. The advanced theories of periodic orbits have a counterpart in synchronous machine theory. The analogue of eccentricity of orbit is internal resistance, while the periodic coefficients of the differential equations introduced by rotating axes in special cases of the three-body problem are introduced in machine theory by the armature rotation. The non-linear equation of damped pendulum motion has an analogue in the study of synchronous machine stability, the hunting of the armature corresponding to pendulum motion. The quasi-differential and integral equations of Cotton are of service in the investigation of locomotive oscillations. Partial differential equations of the eighth order with boundary conditions expressed by differential equations of the fourth order make their appearance in connection with transformer oscillations. Dyadics and tensors have been found of service in machine analysis and stability problems. Lagrange's equations are of importance in mechanical engineering. Nearly every partial differential equation of not more than four independent variables is of some assist-

ance to the engineer. The calculus of variations is of use in many maximum problems. Vector analysis is of use wherever vector fields are found, especially in the discussion of vector magnetic potential within current-carrying regions and in the applications of Maxwell's equations. The Heaviside operational calculus is of almost as great importance in the heat-flow problems of the mechanical engineer as in the circuit problems of the electrical engineer. Statistical methods have been of service in sampling and in the investigation of windage loss in machines. Dimensional analysis has yielded much information where differential equation methods are too complicated. The theorems of Poincaré on analytic differential equations and recent work on non-linear integral equations facilitate the struggle with non-linear problems of various kinds. The theory of functions of a complex variable is of extraordinary use in solving partial differential equations by means of conjugate functions. The theory of functions also puts the operational calculus on a rigorous basis in the simplest way. Practically all the well-known functions of applied mathematics are employed in the solution of problems in elasticity. The foregoing cases are perhaps sufficient to indicate the new importance which mathematics has come to have in engineering work, and they suggest as well the scope of modern engineering mathematics.

The reason for this new importance is clear. Growing complexity in engineering problems has demanded facilities for their solution. Competition in industry tends to render all qualitative applied science, quantitative; complexity arises from the necessity of including more and more factors in the solution. There was a time, for instance, within the present generation when only two defined reactances were used in connection with applied theory of synchronous machines. There are now at least ten in common use, and still others in the theory. Another illustration of increased complexity is the problem of the riding characteristics of an electric locomotive. These characteristics depend simultaneously upon at least twenty-three parameters. The relative importance of each of these factors and its individual effect upon locomotive motion can be determined only by analytical methods. As a final illustration, look back over the long span of the development of applied science in the field of electricity. In Bacon's time, the science of electricity consisted of an amusing collection of seemingly unrelated facts. Today most of the fundamental well-known relations of this science—whether they be regarding the propagation of electromagnetic waves; calculation of space charge; generation, transmission, and use of power; determination of flux; calculation of eddy currents; the solution of potential; the finding of reactance; or the analysis of circuits—are

expressed by mathematical equations. Thus the commanding importance of mathematical methods in modern engineering rests on the fact that they constitute indispensable facilities for dealing with the new complexities in practical problems.

Mathematics facilitates the development of engineering science in a number of ways. In the first place, it does for engineering precisely what it does for each of the other fields mentioned above; it provides better methods of analysis; it is an essential adjunct to *the scientific method*. The latter is as much the method of the modern engineer as it is of the scientist, and it is important enough to warrant our digressing for a moment to discuss it.

As conceived by Francis Bacon some three centuries ago, it consists essentially of four steps:

> 1. Eliminate prejudices; or, in Bacon's words, "At the entrance of every inquiry our first duty is to eradicate any idol by which the judgment may be warped."
> 2. Collect and study the data regarding the given situation.
> 3. Project an hypothesis which, it seems, might rationalize the particular data.
> 4. Test the hypothesis by applying it to the data, or to new data, thus comparing the calculated and actual values of quantities involved. Bacon says, ". . . but from the light of axioms, which having been educed from those particulars by a certain method and rule, shall in their turn point out the way again to new particulars. . . . Our road does not lie on a level, but ascends and descends; first ascending to axioms, then descending to work."

But it may be asked: What has mathematics to do with the scientific method? Bacon gave also the best answer to this question. "*For many parts of nature can neither be invented with sufficient subtility, nor demonstrated with sufficient perspicuity, nor accommodated unto use with sufficient dexterity without the aid and intervening of mathematics.*"

A distinction should be drawn between the immediate objectives of the scientist and the engineer in their use of the scientific method. The one seeks primarily to establish new fundamental laws of science; the other, to predetermine the consequences of established laws in given situations. Yet in either case the approach is the same, and mathematics enters in the same way: induction leads to a generalization; deduction, in turn, leads to predicted particulars. Mathematics enters primarily in the second half of this " ascending " and " descending " process.

A good illustration on the science side is the classic work of Kepler and Newton. Kepler studied Tycho Brahe's data regarding the motion of planets, and was successful in framing an hypothesis that ration-

alized the data. The hypothesis was that during the elliptical motion of a planet the radius vector joining the focus of the ellipse to the planet traversed equal areas in equal intervals of time. This was half of the story. Newton completed it, at least for the time being—that is, until Einstein. Newton carried the generalization further to inquire why planets should thus behave; and after a study of all data at hand projected the generalization that, if heavenly bodies attracted each other in direct proportion to the product of their masses and inversely as the square of their distances apart, a planet would have such a motion as Kepler had found it actually to be. Newton tested and proved the validity of his hypothesis by calculating from it the orbits of the moon and the planets and then comparing these with the observed orbits. Both Kepler and Newton followed the scientific method; they studied the situation, projected a generalization, " descended " mathematically to the particular implications of that generalization, and then compared the calculated results with the actual.

The engineering theorist follows precisely these steps, even if his objective is somewhat different. He collects and studies the data relating to the problem in hand; tentatively settles upon the fundamental law or theory, already established by the scientists, from which, it seems to him, he will be able mathematically to deduce the desired particular relationships between the factors of his problem; sets up the equations accordingly, solves them if he can, and calculates magnitudes; and then he makes tests to determine whether his equations give reliable results and thus also whether they can be depended upon to predict, for instance, the performance of structures or apparatus of new design.

Steinmetz's symbolic method of solving alternating-current circuits affords a good illustration. With the introduction of alternating currents came the then difficult problem of calculating the performance of such circuits. After a study of available electrical data and a review of existing theory which might be useful, he projected the generalization that alternating currents and voltages, being approximately sine-wave phenomena, could be represented by the complex numbers. Applying this generalization, he set up equations representing currents and voltages, calculated their magnitudes and compared these with the actual ones, thus testing and proving the validity of the generalization.

Having reviewed the significance of the scientific method and the relation of mathematics to it, we may now return to the consideration of other respects in which mathematics facilitates the development of engineering science.

Demonstration or clearness of perception is, in certain instances,

obtained by analogy, and a number of these in science are set forth by mathematics. For example, Laplace's equation is satisfied by the following functions: gravitational potential in regions unoccupied by attracting matter, electrostatic potential at points where no charge is present, magnetic potential in regions free from magnetic charges and currents, temperature of an isotropic medium in the steady state, velocity potential at points of a homogeneous non-viscous fluid moving irrotationally, and the real and imaginary parts of an analytic function. Moreover, the behavior of certain electric circuits, on the one hand, and of vibrating mechanical systems, on the other, is represented by the same differential equation.

Mathematics has aided scientific discovery. Recall Maxwell's observation that Ampère's law was inconsistent with the equation of continuity. Maxwell's change in the law led to the electromagnetic theory of light. Thus the validity or non-validity of hypotheses may, in certain cases, be demonstrated by mathematics. One of Heaviside's comments in this connection is interesting. " Faraday," he said, " . . . that great genius had all sorts of original notions wrong as well as right and, not being a mathematician, could not effectively discriminate." And again, Maxwell proved mathematically in 1864 the existence of electromagnetic waves; they were discovered by Hertz in 1887. The " distortionless circuit " was treated by Heaviside in 1892, and the " loaded line," now used extensively in telephone circuits, was devised by Pupin in 1900. In fact, every solution of even an ordinary differential equation is a discovery in the limited sense that it reveals quantitatively how the various quantities are related to each other, even though it should describe no new phenomena.

And, finally, mathematics affords a means of expressing scientific results that is at once compact and accurate. The equation or formula, indicating the numerical relationship between physical quantities in the problems of the engineer, is perhaps the phase of mathematics with which he is most familiar. It affords the means by which the engineering theorist can render easily and accurately available to the engineering practitioner the usable results of engineering science. Without mathematics it would be practically impossible to describe accurately, for instance, the complex relationship between the voltage and current in a long transmission line, between the impressed forces and the magnitude of vibrations in resilient mechanical systems, or between loads and stresses in a beam.

Thus, mathematics facilitates the development and application of engineering science by its being an essential adjunct to the scientific method: in clarifying thinking, correlating and interpreting data,

aiding discovery, setting forth analogies, and expressing scientific results in usable form.

We come now to the educational problem. In the authors' opinion there is one important phase of engineering education which is as yet inadequately developed, and its reasonable development will require a better understanding and a much greater use of mathematics than is now prevalent in the junior and senior years of engineering courses. It is the cultivation of the ability to analyze situations in terms of general principles. Mathematics is, as we have seen, an indispensable facility in such analysis and is thus correspondingly important in the educational process. However, its use in engineering courses is now commonly limited to only two things: utilizing the formula as a means of indicating how certain numerical calculations are to be made; and calling upon the basic concepts of elementary mathematics—trigonometric functions, rates, integrals, etc.—in the study and understanding of the fundamental forms of engineering science. The use of mathematics as a tool in straight thinking is, in the authors' opinion, not at all what it should be.

The reason for this seems clear. It is not, as is sometimes implied, that mathematics has been poorly taught; it is that engineering teachers do not make such use of it after the basic concepts and forms have been reasonably well taught. And the probable reason for this is the limitation of available time. The development of reasoning power requires time. Presumably it has seemed necessary to utilize practically all of the student's available time in his learning (memorizing) the finished results of other minds and in developing certain routine engineering techniques; and thus little time has been left in which his mind could be guided into some independent thinking.

Experience indicates the desirability of a different educational approach, especially for the best minds. In practice, problems are set by situations, machines, and nature, and are not found definitely stated in words or equations. Hence, merely to remember some formulas and the type of problem to which they are applicable is of little avail in solving a new, practical problem. For the latter, not only is a knowledge of the basic engineering sciences necessary, but also a mind disciplined in sound reasoning. In this connection, it is significant that in the General Electric organization fifteen years ago practically all the engineering problems requiring real scientific analysis were referred to a very few individuals, most of whom had received their college training abroad.[1] Today in that organization there are a large

[1] F. C. Pratt, "Professional Engineering Education for the Industries," *Journal of Engineering Education*, Vol. 12, No. 5, January, 1922, pp. 227–235.

number of young men who can solve such problems. The difference is largely that these men, unlike their predecessors at the same age, have been disciplined in sound thinking in the Advanced Course in Engineering of that organization. Moreover, they are exercising leadership not only in the highly technical sides of engineering but also in commercial engineering and in executive capacity. The thing that seems to count professionally is the cultivated intellect. The prominent place of mathematics and physics in the Advanced Course affords not merely a preparation for future specialized theoretical work involving these, but as well a rigorous discipline in analysis—in the art of thought. It thus seems clear that such a discipline should begin more definitely in college than it does at present.

How can this be accomplished, considering the present crowded undergraduate curriculum? Graduate study, which occurs as an answer on first thought, is not a complete solution for two reasons. In the first place, the student might not take graduate work; and secondly, even if he did, it would still be highly desirable to start a more definitely planned development of scientific thinking in undergraduate years in order for him to be properly prepared for graduate study on a genuinely scientific level. Only half of the educational job is done when subject-matter has been mastered; the other half is disciplining the mind in applying the acquired knowledge. To accomplish this, the undergraduate curriculum for the better students should involve a gradual shift of emphasis during the junior and senior years, placing more and more upon scientific thinking and correspondingly less, in point of time, upon further extensions of the student's accumulation of memorized subject-matter. If the engineering graduate can think constructively and independently, he will readily acquire in professional practice the additional knowledge which, from time to time, he needs. However, if he has not developed scientific habits of thought, he will be embarrassed when he faces a real problem, notwithstanding an abundant knowledge of facts and forms. So the plan is to develop an undergraduate course, which will have a logical extension into graduate study, for the specific purpose of building up scientific habits of thought, exercise being obtained in the application of fundamental principles of mathematics, physics, and mechanics to the solution of physical problems. The course would begin where instruction in these subjects commonly ends—at the beginning of the junior year—and extend through the senior year, and also into graduate study, if taken.

The present text is intended to fit into such an educational program, and thus certain observations regarding it should be noted. The first relates to its plan. Such a book must show the engineering student

how to use mathematics—how to reduce the phenomena under observation to mathematical equations, and then how to solve them. The first part of this process is the larger and usually the more difficult; when an engineering problem is reduced to equations, a long step has been taken toward the solution. Although naturally one cannot lay down detailed rules which would make possible the reduction of every engineering problem to equation form, nevertheless one can specify and emphasize the general steps in procedure according to which many problems can be so reduced, and thus aid the development of an orderly habit of thought. The procedure involves largely the subject-matter of physics. The method is outlined in Chapter I.

The second part of the process involves principally the subject-matter of mathematics. The text should explain in a careful manner the mathematical theories and form of most use in engineering. Moreover, it should contain a certain amount of information *about* mathematics. An engineer likes to know the underlying idea and purpose back of a mathematical concept or process before studying in exhaustive detail the process itself, just as a mathematician interested in learning induction-motor theory would appreciate a short general introduction describing the behavior of the machine before taking up the details of some special problem—say the transient performance of the motor after a shock. In the second and subsequent chapters, an attempt has been made to present modern engineering mathematics from the above points of view.

There are other observations which may render the character of the text more intelligible. The engineer's approach to a mathematical concept, function, or theory is different from that of the mathematician. It is not necessary for the engineer, as it is for the mathematician, to acquire an exhaustive knowledge of every mathematical function with which he deals. He needs to know a restricted definition and enough of its functional properties to use it intelligently. For illustration, he is not particularly interested in Bessel functions of fractional orders, although he is in those of positive integral orders. Moreover, the usefulness of the knowledge of the latter would not justify a semester's course in the subject. The engineer has complete confidence in the validity of established mathematical processes. Consequently proofs in themselves are of only secondary interest. Yet a book devoid of proofs is a handbook. Proofs do assist in the understanding and use of mathematics, and hence this text presents a limited number. It is frequently sufficient for the purpose in view to prove a special case of a theorem and then give the general statement. Because of its partial treatment of many functions, the text gives a limited number of

references to fuller treatments of mathematical topics and to proofs omitted.

Often the clearest approach to a mathematical concept, function, or theory is through the solution of an introductory problem, rather than through a formal approach by means of definitions, axioms, and theorems. Greenhill's *Introduction to Elliptic Functions* is a good illustration of this. An engineer is interested primarily in the application of mathematics to the solution of problems, and hence if he sees how some function or theorem is used in a problem he will be better able to understand the use. Thus the device of an introductory problem is utilized in the text.

And finally, undergraduate classes using the text would presumably be led either by an instructor of mathematics or of engineering, depending upon local conditions, although the more advanced work, including subject-matter which the student has not previously studied, would probably be given by an instructor of applied mathematics.

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MATHEMATICS OF MODERN ENGINEERING

CHAPTER I

MATHEMATICAL FORMULATION OF ENGINEERING PROBLEMS

The type of engineering problem which lends itself most readily to mathematical methods of solution is that which involves primarily deductive reasoning. In such a problem the requirement is to determine the implications of some physical law in a given situation. Although of course the solution always involves as well the inductive process of settling upon what law to employ, and the scheme of solution, it is nevertheless primarily the deductive thought which is facilitated by mathematics. The difference between reasoning, on the one hand, step by step from cause to effect, and, on the other, by the use of mathematics is that in the latter case the conclusions from the premises are found largely by manipulation of mathematical symbols according to definite rules. Another difference is that, if the mathematical rules are not violated, one can be assured, as one cannot be in the non-mathematical procedure, that no factor included in the premises has been overlooked in the results; and thus that the results are as true as the premises. Moreover, as problems become more complex by the inclusion of more factors, a stage of complexity is reached beyond which a solution without the aid of mathematics is too difficult even for the best minds.

However, all this is not to say that the non-mathematical procedure is wholly useless or ineffective. On the contrary, it is absolutely essential in its place. It is complementary to the mathematical procedure; without it, mathematics would be useless in engineering. In this field, at least, one cannot extract thought from mathematics without first having put some in. Moreover, the correctness of the solution cannot be tested and its engineering significance appraised without some thought apart from the mathematics. Thought and judgment there must be, and mathematics is an extremely effective tool to facilitate these.

Before mathematics can be thus used the problem must first be formulated in mathematical terms, and it is the purpose of the present chapter to outline the method for this.

1. Deductive Method. There are certain essential, consecutive steps in the deductive method which lead to the mathematical formation of a problem. The first is obviously to define the problem; the second, to settle upon a scheme of solution based upon some fundamental physical law; the third, to state precisely the conditions which, according to that law, must be satisfied; and the fourth, to translate that statement into mathematical form. The entire procedure is illustrated in the formulation of the Introductory Problem, § 6, Chap. II.

Considered in more detail, the first step involves a definite description, including a sketch, of the situation to be analyzed, and a statement of the result desired.

The second step is one which, perhaps even more than the others, requiries a searching survey of one's resources in knowledge and imagination in order to settle upon the most likely scheme of attack. The scheme must include not only the fundamental law which is to serve as the basis of the solution, but also (1) the assumptions which are to be made in order reasonably to simplify the solution, and (2) a plan, however tentative and indefinite, according to which it appears that the desired results can be attained. The plan may, and usually does, involve the subsequent use of additional physical laws relating the quantities in hand to the independent variables. It is not always possible at the start to see clearly what relations will be required in carrying out a projected plan. To determine an effective scheme of attack is an exacting process, and often this object is not accomplished in the first trial.

The third step demands clear and precise thought. It consists in reasoning out and stating in precise English the conditions which must be satisfied in accordance with the physical law settled upon as the basis of the solution. Such a statement constitutes the fundamental premise from which follow all the functional relationships in the solution. Hence it is highly important that the statement should be most carefully and precisely formed. *This statement is indeed the fundamental equation expressed in words.*

The final step in the formulation is the translation of that equation from words to mathematical symbols. All the different terms in the equation are physical quantities of the same kind—the quantities may all be, for instance, forces or energies or momenta—and for each term must be substituted an equivalent mathematical term. Moreover,

this equivalent must be in each case in terms of the chosen independent and dependent variables. For instance, in the simple problem of the motion of a body in a straight line, time is ordinarily taken as the independent variable, the displacement as the dependent variable. In order to determine such mathematical equivalents certain arbitrary conventions must be decided upon.

2. Coordinates. In the first place an appropriate coordinate system must be chosen. This is a difficult step for most students. Indeed, it is not a simple matter, even for graduate students, in the more complicated problems, involving, for example, a system in which there may be two or more mutually dependent motions. The perplexing thing is not so much the concepts of the various forms of coordinates themselves as it is the problem of deciding upon what particular form to use and where to locate it in the given configuration representing the problem. A helpful principle is to select that form of coordinate system—cartesian, spherical, cylindrical, etc.—and to place it in that position which together will yield the simplest mathematical expressions. This process requires the same kind of visualization of possibilities and of their implications as does that of settling upon a general scheme of solution, referred to above. To carry out the process effectively in either case presupposes some experience. At first it may seem difficult not merely to choose the best coordinate system, but even to choose any at all which can be used to formulate the problem. However, a few thoughtful trials will provide some basis for judgment. To give some idea of the result of choosing different coordinates, example (*a*) of § 10, Chap. II, is carried through for two different locations of the origin, a simpler differential equation being obtained in the second case than in the first.

3. Direction. Another arbitrary choice must be made: what direction will be considered as positive? In problems of motion, for instance, two choices must be made. One, which is essentially involved in the coordinate system, is the direction of positive displacement; the other, of positive force. Although the choice in either case is arbitrary, it should be made, like that of the origin of coordinates, so as to retain simplicity in relationships. Then after these choices have been made they must be adhered to rigorously.

In problems of dynamics, forces are expressed as functions of displacement or of time or of both; and thus in order to make correct mathematical substitutions in the word-equation it is necessary to have not only the algebraic form of the term expressing the functional relationship in each instance, but also the proper algebraic sign of the term. The question of algebraic sign causes more *needless* worry in

the mathematical formulation of engineering problems than any other. Needless, simply because careless thinking, or no thinking, has been done in defining positive *direction* before the term is written down; then one wonders what the sign should be! However, with definite criteria established to which the question can be referred, one can proceed rationally.

In simple problems of the motion of a body in a straight line it is customary, although not necessary, to consider force to be positive when it is in the direction of positive displacement. Under such an assumption, the definitions of the derivatives of displacement with respect to time, and the precise statement of the equation in words, together constitute the required criteria for settling the question of algebraic sign. Suppose, for illustration, that one of the forces acting on a body in motion along the x-axis is a "damping force," proportional to the velocity of motion. Such a force *always* opposes the motion, whatever may be the direction of the motion. The magnitude of the force is evidently

$$k\frac{dx}{dt},$$

where k is a coefficient of proportionality. The derivative $\frac{dx}{dt}$ is the time rate of change of displacement. When the motion is in the direction of positive displacement $\frac{dx}{dt}$ is positive. Since we are expressing a directed quantity, force, in terms of another directed quantity, velocity $= \frac{dx}{dt}$, and these two quantities always have opposite senses, there must be a minus sign in front of the term if the positive direction of force is taken as the positive direction of displacement. The term thus becomes

$$-k\frac{dx}{dt}.$$

It may be further emphasized that the minus sign does not mean that the force represented by the term is always directed toward negative displacement. Its sense is that only when $\frac{dx}{dt}$ has a positive sense.[1] In the case of a constant term, however, a negative sign before it does indicate, of course, that its sense is always opposite to the direction

[1] If the positive direction of force had been taken opposite to that of positive displacement, then obviously the algebraic sign of the term would have been plus.

defined as positive for the quantity. The question of the algebraic sign for damping and similar terms is fully illustrated in the Introductory Problem, § 6, Chap. II.

In problems of the electric circuit, the directions of currents and voltages are relative to the physical elements of the circuit, and thus the respective positive directions are defined with reference to such elements.

4. Units and Notation. In order for an equation to indicate correct numerical relationships, either a standard set of units must be used; or, if any units are introduced in a problem that do not belong to the otherwise standard set employed, then a conversion factor must be used to compensate for the numerical difference they introduce. The classical example to illustrate this point is the relation between force f, mass M, and acceleration a. This relation is expressed by the equation

$$f = kMa,$$

where k is a constant depending upon the units assigned to the three quantities. For certain units, k is unity. Obviously, units can be arbitrarily assigned to any two of these three quantities, but if k is to be unity, the unit for the third is thus fixed. Certain standard sets or " systems " of units are in use in which k is unity: when f is in dynes, a in centimeters per second per second, and M in grams; when f is in poundals, a in feet per second per second, and M in pounds mass; when f is in pounds force, a in feet per second per second, and M in slugs.

Or we may look at the matter more fundamentally from the viewpoint of dimensions. It was learned in physics that all quantities relating to motion can be expressed dimensionally in terms of three properly chosen fundamental quantities; and the usual, though not necessary, choice of these is mass, length, and time. After the magnitudes of the three suitable fundamental units have been arbitrarily assigned, the units of other physical quantities are fixed so that constant factors are not required in most of the equations where the quantities are used. Such units comprise a consistent set, three illustrations of which are given above.

Throughout this text the units relating to problems of motion are those based upon the fundamental units:

mass in slugs,
length in feet,
time in seconds.

For illustration, in the physical law referred to above, a force of one pound will produce an acceleration of one foot per second in a mass of one slug.[2] The numerical value for mass is usually determined from the force which gravity exerts upon it, that is, its weight. Thus

$$M = \frac{W}{g},$$

where W is the weight in pounds and g the accleration in feet per second per second due to gravity.

Closely related to the question of units is that of notation. Vague definition of mathematical symbols ranks next to the confusion of algebraic signs in causing the student, and also others, needless worry and errors in the mathematical formulation of a problem. There may be some connections in which a mathematical symbol can represent anything in general, but engineering is not one of them. Clean-cut and precise definition of each symbol, including the unit, is absolutely essential in the formulation, solution, and application of equations in engineering.

[2] A quantitative idea of this unit may be obtained from the fact that the mass of 4 gallons of water is approximately one slug.

CHAPTER II

BASIC ENGINEERING MATHEMATICS

The elementary mathematics of this chapter is basic to the solution of the more usual engineering problems, and is, moreover, fundamental to the study of the more advanced engineering mathematics of subsequent chapters. Most engineering graduates have studied some of the topics treated in the present chapter, but few have a working knowledge of all.

I

ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Section I is concerned with the derivation and solution of ordinary differential equations with constant coefficients by the usual classical methods. Operational methods of handling systems of such equations, and the derivation and solution of the more complicated systems, are considered in later chapters.

5. Point of View. We may distinguish between the engineering and the mathematical points of view. The study of ordinary differential equations, from a mathematical point of view, consists of three parts:

(*a*) A proof of the existence of a solution of a single equation or system of equations.

(*b*) The investigation of the properties of the solution: continuity, differentiability, analyticity, and integrability of the solution with respect to both the independent variable and important parameters present.

(*c*) The construction of a solution in a form suitable for the use at hand.

The study of differential equations from an engineering point of view is different. It consists mainly of but two parts:

(*a*) The derivation of the differential equation.

(*b*) The solution of the differential equation.

The existence of a solution is taken for granted from physical considerations and from the assumption that the differential equation represents the physical problem under consideration. The engineering

point of view will be adhered to, especially in the treatment of this section.

6. Introductory Problem. A simple problem may facilitate the understanding of differential equations. Following the approach outlined in Chap. I, let us suppose that a body of mass M, constrained to move in a straight line, is acted upon by a spring as shown in Fig. 1. The spring may be both extended and compressed, and always acts on the body in the line of displacement. The motion takes place without friction. At the beginning of the period under consideration, when time $t = 0$, the body has such a position that the spring is undistorted, and is moving to the right with velocity v. It is desired to determine the equation of motion of the body, that is, its position at any time t subsequent to $t = 0$.

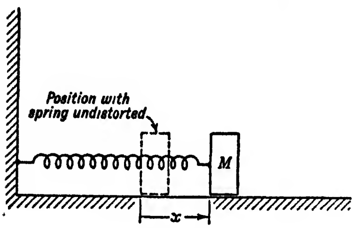


FIG. 1.

The solution may be based upon Newton's[1] second law of motion. According to that law the condition that must be satisfied is: the force applied to the body is equal to the product of its mass and acceleration, and the acceleration takes place in the direction of the applied force. This is the equation, stated in words, which later will be translated into mathematical terms.

This requires the selection of a coordinate to designate the position of the body. Let x be the displacement of the body from such a position that the spring is undistorted, x being positive when the body is to the right of this position. The velocity is then $\frac{dx}{dt}$; the acceleration, $\frac{d^2x}{dt^2}$. In accordance with the definition of the derivative as a rate of *change*, $\frac{dx}{dt}$ is positive when the body moves to the right, negative when it moves to the left; and $\frac{d^2x}{dt^2}$ is positive when the acceleration is directed to the right, negative when it is directed to the left. Similarly, as discussed in Chap. I, the algebraic sign of the applied force denotes its direction. Let the convention for the sign of force be the same as for that of displacement, positive to the right, negative to the left. The spring exerts a force f, which is in the direction oppo-

[1] Newton's laws of motion are stated in § 9.

site to the displacement, and, by Hooke's law, is proportional to the displacement. Thus

$$f_s = -kx,$$

where k is a coefficient which is constant if the elastic limit of the spring is not exceeded.

Thus the previous equation stated in words becomes

$$M\frac{d^2x}{dt^2} = -kx,$$

or

$$\frac{d^2x}{dt^2} + \frac{k}{M}x = 0. \tag{1}$$

Eq. (1) is called the the **differential equation of motion of the body.** The solution, as determined by methods described in later sections, is

$$x = A\sin\sqrt{\frac{k}{M}}t + B\cos\sqrt{\frac{k}{M}}t, \tag{2}$$

where A and B are arbitrary constants. That this is a solution may be verified by its substitution in Eq. (1).

The arbitrary constants A and B are called **constants of integration.** The conditions $x = 0$ and $\frac{dx}{dt} = v$ at $t = 0$, given in the statement of the problem, are called **initial conditions.** By means of the initial conditions particular values of A and B can be determined so that (2) will give the position of the body at any time t subsequent to $t = 0$. If (2) be differentiated with respect to time and then the conditions $x = 0$, $\frac{dx}{dt} = v$, and $t = 0$ be substituted in both (2) and the derivative of (2) there results

$$\left.\begin{aligned} 0 &= A\sin\sqrt{\frac{k}{M}}0 + B\cos\sqrt{\frac{k}{M}}0, \\ v &= A\sqrt{\frac{k}{M}}\cos\sqrt{\frac{k}{M}}0 - B\sqrt{\frac{k}{M}}\sin\sqrt{\frac{k}{M}}0, \end{aligned}\right\} \tag{3}$$

which may be solved for A and B, to obtain

$$A = v\sqrt{\frac{M}{k}}, \quad B = 0.$$

Substitution of these values in (2) gives

$$x = v\sqrt{\frac{M}{k}} \sin \sqrt{\frac{k}{M}}\, t. \tag{4}$$

Eq. (4) is the desired **equation of motion,** the answer to the problem stated. It is valid for the units given in § 9. Reference to the above introductory problem will serve to illuminate some of the essential aspects of differential equations.

7. Origin of Some Differential Equations. In the mathematical formulation of a problem, as discussed in Chap. I, the third step is a statement of the equation in words. If any of the quantities related in that statement involve rates of change, the mathematical equivalents are written in terms of derivatives and the resulting symbolic equation is a differential equation. In the example of the previous article one of the quantities related is the acceleration of the body, which in symbolic form is $\frac{d^2x}{dt^2}$. Thus Eq. (1) is a differential equation.

Other physical quantities that involve rates and therefore lead to differential equations are numerous. For example, in geometric problems slope and curvature depend upon derivatives. In electric-circuit theory, current is the rate of change of a charge, and induced voltage is proportional to the rate of change of magnetic flux linkages.

8. Definitions. The following general definitions relate to differential equations. An equation involving derivatives or differentials is a **differential equation.** An **ordinary differential equation** is one containing only ordinary or total derivatives as distinct from partial derivatives. The **order** of a differential equation is the order of the highest derivative which the equation contains. For example, Eq. (1) is of second order. The **degree** of a differential equation is the degree of the highest-ordered derivative which the equation contains after it has been cleared of radicals and fractions with respect to all derivatives. Eq. (1) is of first degree. A **solution** of an ordinary differential equation is a relation between the dependent and independent variables which satisfies the differential equation. A solution is called the **general solution** if it contains a number of arbitrary constants equal to the order of the equation, provided the constants enter into the solution in such a way that they cannot be replaced by a smaller number of equivalent arbitrary constants. Eq. (2) is the general solution of Eq. (1). A **particular solution** of a differential equation is a solution obtained from the general solution by giving particular values to one or more of the arbitrary constants. Eq. (4) is a particular

solution of Eq. (1). If the differential equation is of order n and the value of the dependent variable and the values of all derivatives up to and including those of order $n - 1$ are known for a particular value of the independent variable, then the n arbitrary constants of the general solution can be determined. These values of the dependent variable and of the derivatives up to and including those of order $n - 1$ are called **initial conditions** or **boundary conditions.** In Eq. (1) $n = 2$, and the two Eqs. (3), which are statements of the initial conditions, were used to determine the constants.

9. Principles of Mechanics. The physical principles with which we are first concerned fall into two groups; those pertaining to mechanics and those relating to elementary electrical phenomena. Since mechanical forces and motions are, in general, more easily visualized than electrical ones, we begin with the derivation and solution of the differential equations of some simple mechanical systems. Following this we shall consider the formulation of the differential equations of simple electric circuits. Since these equations are of the same type as the ones for mechanical systems, the solutions are of the same form and need not be discussed again. Such mechanical principles as are used in Sec. I are briefly treated in this article.

The whole science of kinetics rests upon Newton's laws of motion. These experimental laws may be stated as follows:[2]

1. A particle under the action of no forces remains at rest or moves in a straight line with constant speed.
2. The resultant force acting on a particle equals in magnitude and direction the product of its mass by its acceleration.
3. If two particles exert forces on each other, the force exerted by the first on the second (action) is equal and opposite to the force exerted by the second on the first (reaction).

The first law is but a special case of the second. These two laws hold only with respect to a non-accelerated frame of reference, which, in most engineering problems, may be a frame that is fixed, or moving with a constant velocity, relative to the earth.

The third law states that forces always occur in *pairs* and that the forces of a pair are equal in magnitude but act in opposite directions.

The laws of motion for a particle may be extended to the case of a rigid body, which is nothing more than a group of particles. It is found that the resultant vector force is equal the product of the mass by the vector acceleration of the center of mass of the body. It is convenient, however, to employ a different statement of this law.

[2] Page, *Introduction to Theoretical Physics*, courtesy of D. Van Nostrand Co., Inc.

Suppose that a number of forces, $f_1, f_2, \ldots$, act upon a body free to move in translation only. These forces may all be added vectorially to form a resultant which is the equivalent of the applied forces. Symbolically

$$f = \sum f_i,$$

where $\sum f_i$ represents the vector sum of $f_1, f_2, \ldots$. If M is the mass of the body, and a is the acceleration (which has the direction of the resultant force), then by the second law

$$f = Ma,$$

or

$$\sum f_i = Ma.$$

Thus

$$- Ma + \sum f_i = - Ma + f_1 + f_2 + \ldots = 0. \tag{5}$$

Now if the body were in static equilibrium the condition

$$\sum f_i = 0 \tag{6}$$

would hold. This condition may be generalized to hold for any state of rest or translation by regarding $- Ma$ in (5) as one of the forces f_i in (6) acting on the body. Treating the equivalent force $- Ma$, called the inertial reaction, as one of the applied forces in (6) is in accordance with D'Alembert's principle.[3] These equations hold for the forces and acceleration considered as vectors, and therefore also for the algebraic components in any direction.

Besides these laws relating applied forces to inertial reaction, it is frequently necessary to employ laws relating certain of the forces other than inertial reaction to displacement and its derivatives. For example, in the introductory problem of § 6 the relation

$$f_s = - kx$$

between the resilient force f_s due to the spring, and the displacement x, was used. This is a statement of Hooke's law for elastic distortion. Another of frequent occurrence in mechanical problems is the approximate relation between viscous damping force f_d and rate of change of displacement, $\dfrac{dx}{dt}$,

$$f_d = - k_d \frac{dx}{dt},$$

[3] It is, in fact, D'Alembert's principle for the limited case of a single body with a single degree of freedom.

where k_d is the damping coefficient, usually taken as constant. Such a damping force might be exerted by a shock-absorber or dash-pot.

The negative signs in the above expressions for inertial reaction, damping, and resilient forces on a body in a simple system are a consequence of the physical fact that these forces all *oppose* its acceleration, velocity, and displacement, respectively, and that the positive direction of force is the same as that of displacement.

There may be torques on a rigid body in rotation similar to the forces on one in translation. The relation between torque and angular acceleration may be developed by applying Newton's laws to elementary divisions of mass, regarded as particles, and then summing the moments of the forces on the particles. Let T denote resultant torque applied to the body, I its moment of inertia, θ its angular displacement, $\frac{d\theta}{dt}$ its angular velocity, and $\frac{d^2\theta}{dt^2}$ its angular acceleration, and there is obtained

$$T = I\frac{d^2\theta}{dt^2},$$

which is analogous to $f = ma$ or $f = m\frac{d^2x}{dt^2}$.

T is the resultant torque, in general the sum of torques $T_1, T_2, \ldots,$ or ΣT_i. In accordance with a generalization of D'Alembert's principle we may write

$$\Sigma T_i = 0$$

by including as one torque T_i, the term $-I\frac{d^2\theta}{dt^2}$, which is the inertial reaction of rotation. Also, in any system there may be a damping device that applies a torque

$$T_d = -K_d\frac{d\theta}{dt},$$

and there may be an elastic constraint that applies a torque

$$T_s = -K\theta,$$

where K_d and K are coefficients taken as constants. Although the second relation is quite accurate within limited displacements that frequently occur, the former is likely to be only an approximate law, of particular value because of its simplicity.

The above relations may be expressed numerically in any consistent system of units. A system of mechanical units commonly employed

by engineers is given in the following table, where, for convenience, are also listed the corresponding symbols used in this text. The equations that appear later hold in any consistent system of units, and, in particular, for the ones of the table.

QUANTITY	SYMBOL	UNIT
Time	t	second
Distance	x, y, s	foot
Velocity	$V, \frac{dx}{dt}$	foot per second
Acceleration	$a, \frac{d^2x}{dt^2}$	foot per second per second
Acceleration of gravity	g	32.2 ft. per sec. per sec.
Force	f	pound
Weight (force of gravity)	W	pound
Mass*	M	slug
Spring constant	k	pound per foot
Damping constant	k_d	pound per foot per second
Angular displacement	θ	radian
Angular velocity	$\omega, \frac{d\theta}{dt}$	radian per second
Angular acceleration	$\alpha, \frac{d^2\theta}{dt^2}$	radian per second per second
Torque	T	pound-foot
Moment of inertia	I	slug-foot^2
Rotational spring constant	K	pound-foot per radian
Rotational damping constant	K_d	pound-foot per radian per second

* Mass in slugs $= \dfrac{\text{Weight in pounds}}{\text{Acceleration of gravity, 32.2 ft. per sec. per sec.}}$

10. Derivation of Differential Equations of Simple Mechanical Oscillations. The method described in Chap. I is utilized in deriving the differential equations of motion of the following examples, which are similar in character to the one of § 6 but include more terms. These examples will illustrate the application of the principles of the previous article.

(*a*) *Spring, mass, gravity, and damping vane.* The configuration of Fig. 1, § 6, is turned 90° so that motion occurs in the vertical direction. That is, a body of mass M is suspended by a spring from a fixed support, and is constrained to move only in a vertical line passing through the point of support. See Fig. 2. A viscous damping force is applied by vanes attached to the body. It is given an initial displacement below its equilibrium position of rest and is then released. Let us determine the differential equation describing the subsequent

motion when there are no external applied forces except gravity. The mass of the spring is assumed to be negligible.

D'Alembert's principle will be applied to the problem. Accordingly, the algebraic sum of the forces applied to the body is zero. The forces comprise those due to inertial reaction, damping, resilience, and gravity. Let x be a coordinate measuring displacement down

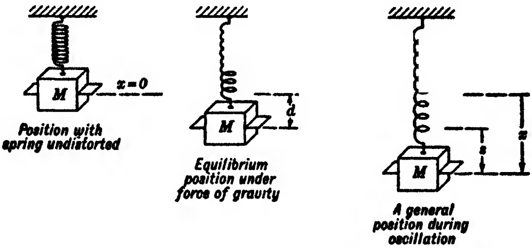


FIG. 2.

from the position where the spring is undistorted, and let the positive direction for force also be down. Then:

The inertial reaction is $-M\dfrac{d^2x}{dt^2}$.

The damping force is $-k_d\dfrac{dx}{dt}$ (approximately).

The resilience force is $-k\,x$ for displacements within the elastic limit.

The gravitational force, equal to the weight, is Mg.

The equation of motion may then be written,

$$M\frac{d^2x}{dt^2} + k_d\frac{dx}{dt} + kx - Mg = 0. \tag{7}$$

The equation may be given a simpler form by the choice of a slightly different coordinate. Let the displacement be measured down as before, but instead of taking the origin as the position where the spring is undistorted, let it be the position of equilibrium under the force of gravity alone. Let the new coordinate be s, see Fig. 2, and let d represent the distance between old and new origins. This distance is the displacement required to give sufficient spring force to balance the gravitational force. Or

$$kd = f_g = Mg. \tag{8}$$

The sum of the forces may again be set equal to zero. The forces are of the same nature as before and may be written:

$$\text{The inertial reaction} = -M\frac{d^2s}{dt^2}.$$

$$\text{The damping force} = -k_d\frac{ds}{dt}.$$

$$\text{The spring force} = -k(s+d).$$

$$\text{The gravitational force} = Mg.$$

Accordingly

$$-M\frac{d^2s}{dt^2} - k_d\frac{ds}{dt} - ks - kd + Mg = 0. \tag{9}$$

By (8), the last two terms cancel each other, and, after multiplication by -1,

$$M\frac{d^2s}{dt^2} + k_d\frac{ds}{dt} + ks = 0. \tag{10}$$

Eq. (10) has one less term than Eq. (7), a simplification which is due to the choice of coordinate. It is true in general that if the coordinate is taken as zero when the body is in equilibrium, instead of when the spring is undistorted, the terms representing steady applied forces such as gravity disappear from the differential equation.

Suppose that it is desired to substitute numerical values for the constants when the data are given as follows. The weight of the body is 20 lb., and under the action of gravity alone, the body has an equilibrium displacement of 4 in. from its position corresponding to the undistorted spring. The damping force in pounds exerted by the vanes is numerically equal to twice the velocity of the body. The weight is pulled down 6 in. below its equilibrium position and then released. From these data

$$M = \frac{20}{32.2} \text{ slugs},$$

$$k_d = 2 \text{ lb. per ft. per sec.},$$

$$k_s = 20 \div \frac{4}{12} = 60 \text{ lb. per ft.}$$

Thus the equation may be written

$$\frac{20}{32.2}\frac{d^2s}{dt^2} + 2\frac{ds}{dt} + 60s = 0. \tag{11}$$

The initial displacement which is 0.5 ft. does not enter into the differential equation of motion, but it does enter into the solution, as explained later.

(*b*) *Torques applied to a rotating system.* A wheel and shaft of combined moment of inertia I are mounted in bearings of negligible friction with the axis in a horizontal position. A spiral spring mounted as shown in Fig. 3 provides an elastic restoring force which opposes the angular displacement of the wheel. Damping vanes are attached to it, and also two forces f_1 and f_2 are applied tangentially to its rim at a distance r from the axis, as shown. It is desired to determine the differential equation of motion of the wheel.

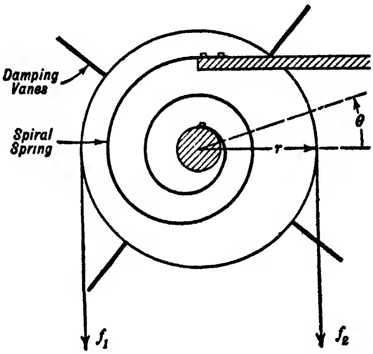


FIG. 3.

According to D'Alembert's principle, the sum of the torques about the axis—due to the inertial reaction, the spring, the damping vanes, and the applied forces—is zero. Let the positive direction for torque and displacement be counterclockwise, and let θ be the angular displacement of the wheel from the position where the spring is undistorted. Then:

The inertial reaction is $-I\dfrac{d^2\theta}{dt^2}$.

The damping vane torque is $-K_d\dfrac{d\theta}{dt}$ (approximately).

The resilience torque is $-K\theta$, for displacements within the elastic limit.

The torque due to f_1 is rf_1.

The torque due to f_2 is $-rf_2$.

Thus, translating into mathematical form the previous equation in words,

$$I\frac{d^2\theta}{dt^2} + K_d\frac{d\theta}{dt} + K\theta - rf_1 + rf_2 = 0, \tag{12}$$

which is the desired differential equation of motion.

(*c*) *Compound system, free vibrations.* Forced vibrations of a system are those caused by a periodic force applied to it; free vibrations, on

the other hand, are characteristic of the system itself and may occur when there is a transient disturbance of its equilibrium. Let us consider a case of free vibrations of the system shown in Fig. 4, consisting of two masses and two resilient members, and constituting a body elastically mounted over a single wheel in much the same way, for example, that an automobile body is mounted over four wheels.

Mass M_1 is supported by a wheel and elastic tire, and mass M_2 is supported above M_1 by a spring. Constraints not shown permit vertical motion only, and the wheel is not allowed to rotate. The

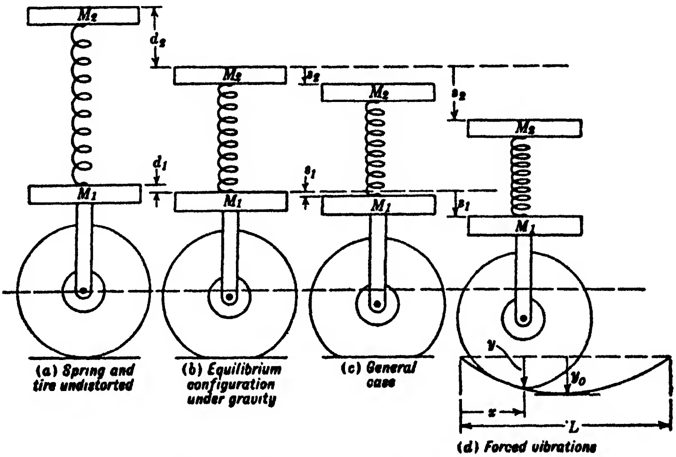


FIG. 4.—Coupled Oscillating System.

masses of all parts except M_1 and M_2 are negligible, as are frictional forces.

Mass M_2 is given a downward displacement and then released. Until M_2 is released M_1 is held in the equilibrium position it would have if undisturbed. It is desired to determine the differential equation describing the subsequent motions of the masses.

D'Alembert's principle will be applied to each of the masses. Accordingly, the algebraic sum of the forces on each mass is equal to zero. The forces on M_2 include its inertial reaction f_{M_2}, the force of the spring f_{S_2}, and the force of gravity f_{G_2}. The forces on M_1 include its inertial reaction f_{M_1}, the force of the spring f_{S_1}, the force of the tire f_{T_1}, and the force of gravity f_{G_1}. Then

$$\begin{aligned} f_{M_1} + f_{S_1} + f_{T_1} + f_{G_1} &= 0, \\ f_{M_2} + f_{S_2} + f_{G_2} &= 0. \end{aligned} \tag{13}$$

Let displacements and forces be taken as positive when directed down. In example (*a*) of this section it was found that simpler equations resulted if the origin of displacement was taken at the equilibrium position. Therefore take the origins of the displacements s_1 and s_2 of masses M_1 and M_2 at the equilibrium positions. (See Fig. 4*b* and *c*.) The distances of these equilibrium positions from the positions of undistorted elastic members will be denoted by d_1 and d_2 respectively, as shown in Fig. 4*a* and *b*.

The inertial reactions are

$$f_{M_1} = -M_1 \frac{d^2 s_1}{dt^2},$$

$$f_{M_2} = -M_2 \frac{d^2 s_2}{dt^2}.$$

The spring forces f_{S_1} and f_{S_2} are

$$f_{S_1} = -k_2 e,$$

$$f_{S_2} = k_2 e,$$

where k_2 is the elastic coefficient of the spring, and where e is its elongation, given by

$$e = d_1 + s_1 - d_2 - s_2.$$

Thus

$$f_{S_1} = k_2(d_2 + s_2 - d_1 - s_1),$$

$$f_{S_2} = -k_2(d_2 + s_2 - d_1 - s_1).$$

Similarly, if for sufficiently small deformations, the tire conforms to the same sort of elastic law as the spring

$$f_{T_1} = -k_1(d_1 + s_1),$$

where k_1 is the elastic coefficient of the tire.

The forces of gravity are

$$f_{G_1} = M_1 g,$$

$$f_{G_2} = M_2 g.$$

Thus Eq. (13) may be written

$$\left.\begin{aligned} -M_1 \frac{d^2 s_1}{dt^2} + k_2(d_2 + s_2 - d_1 - s_1) - k_1(d_1 + s_1) + M_1 g &= 0, \\ -M_2 \frac{d^2 s_2}{dt^2} - k_2(d_2 + s_2 - d_1 - s_1) + M_2 g &= 0. \end{aligned}\right\} \quad (14)$$

These may be simplified by using the relations between the forces of gravity on M_1 and M_2 and the equilibrium displacements. The conditions of equilibrium are that with the bodies at rest, when s_1 and s_2 are zero, the sum of the gravitational and spring forces on each body is zero. Or

$$\left.\begin{aligned} k_2(d_2 - d_1) - k_1 d_1 + M_1 g &= 0, \\ - k_2(d_2 - d_1) + M_2 g &= 0. \end{aligned}\right\} \tag{15}$$

Eqs. (15) determine d_1 and d_2 in terms of the other parameters. Incidentally, (15) may be written by setting s_1, s_2, $\frac{d^2s_1}{dt^2}$, $\frac{d^2s_2}{dt^2}$ each equal to zero in Eq. (14), since these substitutions transform (14), the general equations of dynamic equilibrium, to (15), the equations for the particular case of static equilibrium. Returning to the simplification of (14) we find that certain terms cancel each other as given by (15), and leave,

$$\left.\begin{aligned} M_1 \frac{d^2s_1}{dt^2} - k_2(s_2 - s_1) + k_1 s_1 &= 0, \\ M_2 \frac{d^2s_2}{dt^2} + k_2(s_2 - s_1) &= 0, \end{aligned}\right\} \tag{16}$$

which are the desired differential equations of motion.

(*d*) *Compound system, forced vibrations.* Suppose now that the wheel turns and the whole system has a constant horizontal component of velocity, V. As shown in Fig. 4*d* the tire runs to the right over a road with a series of regular bumps, taken as sinusoidal in cross-section. It is desired to determine the differential equations of motion on the assumption that the tire remains always in contact with the road.

This case is very similar to the previous one. Again D'Alembert's principle is employed, and the same kinds of forces are involved as before.

Since the laws of motion upon which this solution depends hold only with respect to a non-accelerated frame of reference, the origins of our coordinate system must either be stationary or move with a constant velocity. Let us employ the same coordinates s_1 and s_2 as before, permitting the origins to move in the horizontal direction of motion with the constant velocity V.

The expressions for the inertial reactions, the gravitational forces, and the force of the spring are exactly the same as before, but that for the force f_{T_1} of the tire is different. The compression of the tire is

now $d_1 + s_1 - y$ instead of $d_1 + s_1$, if y is the vertical distance of the road surface below its average level. Thus

$$f_{T_1} = -k_1(d_1 + s_1 - y).$$

The quantity y can be expressed as a function of time. Suppose that the sinusoidal ridges and troughs have an amplitude y_0 above and below the average level, as shown in Fig. 4*d*, a half wave length L, and when time $t = 0$, $y = 0$ and the tire is moving downhill. Then

$$y = y_0 \sin \frac{\pi V t}{L},$$

$$f_{T_1} = -k_1\left(d_1 + s_1 - y_0 \sin \frac{\pi V t}{L}\right).$$

Instead of Eq. (16), then, we obtain as differential equations of motion for the case at hand

$$\left.\begin{aligned} M_1 \frac{d^2 s_1}{dt^2} - k_2(s_2 - s_1) + k_1 s_1 &= k_1 y_0 \sin \frac{\pi V t}{L}, \\ M_2 \frac{d^2 s_2}{dt^2} + k_2(s_2 - s_1) &= 0. \end{aligned}\right\} \quad (17)$$

PROBLEMS

1. A mass m is free to move along the x-axis. It is acted upon by a force whose magnitude is proportional to the distance of the mass from the origin and whose direction is *away* from the origin. Write the differential equation of its motion.

2. Write the differential equation for the oscillation of a simple pendulum of length l, mass m, and angular displacement θ from the vertical.

3. A uniform circular disc of moment of inertia I is supported by a vertical elastic rod as shown in Fig. 5. When the disc is turned about its axis, the resilient torque of the rod is k times the angular displacement θ of the disc. The disc is turned through an angle θ_0 and released. Write the differential equation of motion of the disc.

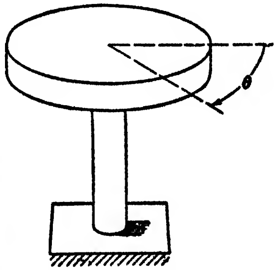


FIG. 5.

4. A cylindrical buoy floats in fresh water with its axis always vertical. The length, radius, and weight of the buoy are respectively l, r, and W. It is depressed until its upper surface coincides with the surface of the water and is then released. Write the differential equations of motion of the buoy on the assumption that the water exerts on it only hydrostatic pressure.

5. A shock-absorber (dash-pot) which acts equally for either direction of motion of its piston is placed in parallel with the spring of Fig. 4. The force exerted by the shock-absorber is always proportional to the difference of the velocities of M_1 and M_2. Let M_2 be depressed while M_1 is maintained in its original equilibrium position and then both released. Write the differential equations of motion.

6. Write the differential equations of the small vibrations of the double pendulum of Fig. 6. The bobs have masses m_1 and m_2, and the strings have lengths a and b. Assume that there is no damping. By small vibrations are meant oscillations so small that for either θ_1 or θ_2, $\sin \theta \doteq \theta$ and $\cos \theta \doteq 1$.

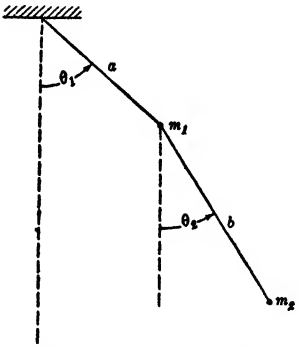


FIG. 6.—Double Pendulum.

11. Solution of Homogeneous Linear Differential Equations with Constant Coefficients.

A homogeneous linear differential equation is one of the form

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \qquad + a_{n-1} \frac{dy}{dt} + a_n y = 0 \tag{18}$$

which contains no term independent of y. If $p^s y$, where $s = 1,2,3, \ldots n$, be written for $\frac{d^s y}{dt^s}$, Eq. (18) is more simply written

$$p^n y + a_1 p^{n-1} y + \ldots + a_{n-1} p y + a_n y = 0. \tag{19}$$

We shall lead up to the solution of Eq. (19) by first solving some particular examples. To solve a differential equation means to obtain the general solution.

EXAMPLE 1. Solve the equation

$$p^2 y + a_1 p y + a_2 y = 0. \tag{20}$$

At this point we encounter a departure from the straightforward methods of solution which characterize the techniques of elementary mathematics. It becomes necessary to assume a solution and then determine its correctness by substitution. Inspection of Eq. (20) shows that it might be satisfied by a function y if the derivatives $p^2 y$ and py were proportional to the function y itself. The derivatives of the exponential function $y = e^{mt}$, where m is a constant, have this property, hence we are led to try e^{mt} as a solution, leaving the value of m undetermined for the present. Substitution of this value for y in Eq. (20) gives

$$e^{mt}(m^2 + a_1 m + a_2) = 0.$$

Since e^{mt} cannot be zero for finite values of t, the equation is satisfied only if the quantity within parentheses is zero, which means that m must have a definite value depending on the constants a_1 and a_2 of the differential equation. If it does have this value, e^{mt} is a solution of the differential equation (20). The equation

$$m^2 + a_1 m + a_2 = 0 \tag{21}$$

has two roots, which for some values of a_1 and a_2 are real, and for others, complex. The assumed solution e^{mt} thus satisfies (20) if m be assigned the value of either of the two roots of (21). Moreover, if the solution is multiplied by a constant, which may be either real or complex, the resulting product is still a solution of (20) as may be shown by substitution. That is, both $y = C_1 e^{m_1 t}$ and $y = C_2 e^{m_2 t}$ satisfy (20) if m_1 and m_2 are the roots of (21) and C_1 and C_2 are any real or complex constants. It may also be shown by substitution that the sum

$$y = C_1 e^{m_1 t} + C_2 e^{m_2 t} \tag{22}$$

satisfies the differential equation (20). If C_1 and C_2 are independent arbitrary constants, (22) is the general solution, by the definition of § 8. The two constants in Eq. (22) are independent if they cannot be combined into a single equivalent arbitrary constant. Such a combination can be made only if $m_1 = m_2$ (See Eq. (30)), and then the solution is not the general solution.

Eq. (21) is called the **auxiliary equation** or **characteristic equation** of (20). When the roots m_1 and m_2 of (21) are real, (22) is a convenient form for the solution. When the roots are complex, a different form is more useful. If the roots are equal, (22) is no longer the general solution, and a different form of function must be employed. The cases of complex and equal roots of the auxiliary equation will be treated in the following examples.

Example 2. Solve the equation

$$p^2 y + a_1 p y + a_2 y = 0, \tag{23}$$

where the roots of the auxiliary equation $m^2 + a_1 m + a_2 = 0$ are complex. If the roots of the auxiliary equation are[4] $a \pm b\sqrt{-1} = a \pm bi$, then the general solution of (23), from example 1, is

$$y = C_1 e^{(a+bi)t} + C_2 e^{(a-bi)t} = e^{at}\left[C_1 e^{ibt} + C_2 e^{-ibt}\right]. \tag{24}$$

[4] Throughout this chapter the symbol i is used to designate $\sqrt{-1}$.

It may be shown as follows that Eq. (24) may be written

$$y = e^{at}(A \sin bt + B \cos bt), \tag{25}$$

where only real quantities are present. Since

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots,$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots,$$

and

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots.$$

for z real or complex, it follows that

$$\begin{aligned} e^{ibt} &= 1 + ibt + \frac{(ibt)^2}{2!} + \frac{(ibt)^3}{3!} + \frac{(ibt)^4}{4!} + \dots \\ &= \left(1 - \frac{b^2t^2}{2!} + \frac{b^4t^4}{4!} - \dots\right) + i\left(bt - \frac{b^3t^3}{3!} + \frac{b^5t^5}{5!} - \dots\right) \\ &= \cos bt + i \sin bt. \end{aligned} \tag{26}$$

Likewise

$$e^{-ibt} = \cos bt - i \sin bt. \tag{27}$$

By the substitution of (26) and (27) in (24), the last becomes

$$y = e^{at}\,[(C_1 - C_2)i \sin bt + (C_1 + C_2) \cos bt]. \tag{28}$$

If y is to be a real quantity, $(C_1 - C_2)i$ and $(C_1 + C_2)$ must be real. This implies that C_1 and C_2 are conjugate. If $C_1 = \frac{B - Ai}{2}$ and $C_2 = \frac{B + Ai}{2}$, where A and B are real, Eq. (28) reduces to (25), and still has two arbitrary independent constants.

If the roots of the characteristic equation are pure imaginaries then $a = 0$ in Eqs. (25) and (28). Eq. (2) of the introductory problem is of this form and can be solved by the method of the present section.

Example 3. Solve the equation

$$p^2y - 2\,m_1py + m_1^2y = 0. \tag{29}$$

The auxiliary equation

$$m^2 - 2\,m_1m + m_1^2 = 0$$

has the double root $m = m_1$. By example 1, the solution is

$$y = C_1e^{m_1t} + C_2e^{m_1t} = Ae^{m_1t}. \tag{30}$$

But (30) contains only one arbitrary constant, namely, $C_1 + C_2 = A$. Thus (30) is not the general solution since by definition the solution of a second-order differential equation must contain two independent arbitrary constants. However, it can be verified by substitution that

$$y = (A + Bt)e^{m_1t}$$

satisfies the differential equation (29) and, since it contains two arbitrary constants, it is the general solution.[5]

GENERAL EQUATION. We now return to the solution of differential equation (19). The principles employed in the last three examples are adequate for the solution of this general equation. By substituting $y = e^{mt}$ in Eq. (19), the characteristic equation is found to be

$$m^n + a_1m^{n-1} + \ldots + a_n = 0. \tag{31}$$

If the n roots of Eq. (31) are distinct, then the general solution of (19) is

$$y = C_1e^{m_1t} + C_2e^{m_2t} + \ldots + C_ne^{m_nt}. \tag{32}$$

Such complex roots as occur always occur in conjugate pairs. If Eq. (31) has only one pair of complex roots (say $a \pm bi$), then (32) may be reduced, by Eqs. (26) and (27), to the form

$$y = e^{at}(A \sin bt + B \cos bt) + C_3e^{m_3t} + \ldots + C_ne^{m_nt}. \tag{33}$$

Similarly, if there are two pairs of complex roots $(a_1 \pm b_1i)$ and $(a_2 \pm b_2i)$, the roots all being distinct, (32) may be reduced to the form

$$\begin{aligned} y = e^{a_1t}(A_1 \sin b_1t + B_1 \cos b_1t) &+ e^{a_2t}(A_2 \sin b_2t + B_2 \cos b_2t) \\ &+ C_5e^{m_5t} + \ldots C_ne^{m_nt}. \end{aligned}$$

If r of the n roots of (31) are equal (say $m_1 = m_2 = \ldots = m_r$), then it is verified by substitution[6] that the general solution is

$$y = (C_1 + C_2t + \ldots + C_rt^{r-1})e^{m_1t} + C_{r+1}e^{m_{r+1}t} + \ldots + C_ne^{m_nt}. \tag{34}$$

Finally suppose that one pair of complex roots $(a \pm bi)$ occurs r times. Then Eq. (34) becomes

$$\begin{aligned} y &= (C_1 + C_2t + \ldots + C_rt^{r-1})e^{(a+bi)t} \\ &+ (D_1 + D_2t + \ldots + D_rt^{r-1})e^{(a-bi)t} \\ &+ C_{2r+1}e^{m_{2r+1}t} + \ldots + C_ne^{m_nt}. \end{aligned}$$

[5] See also § 16. [6] See § 16.

This equation, by means of Eqs. (26) and (27), can be reduced to the form

$$y = e^{at}[(A_1 + A_2t + \ldots + A_rt^{r-1}) \sin bt$$
$$+ (B_1 + B_2t + \ldots + B_rt^{r-1}) \cos bt]$$
$$+ C_{2r+1}e^{m_{2r+1}t} + \ldots + C_ne^{m_nt}. \tag{35}$$

The equations characterizing free vibrations of a system usually are of the form (19), or may be reduced to that form. For example, (10), (11), (12), and (16) are such equations. On the other hand, forced vibrations, arising in a system to which a periodic force is applied, are characterized by non-homogeneous differential equations.

12. First Method of Solution of Non-homogeneous Linear Differential Equations. Reduction by Differentiation to Homogeneous Form. A non-homogeneous linear differential equation is one of the form

$$p^ny + a_1p^{n-1}y + \ldots + a_{n-1}py + a_ny = f(t) \text{ (or a constant)} \tag{36}$$

which thus contains a term independent of y.

We lead up to the solution of (36) by first solving particular examples.

EXAMPLE 1. Obtain the solution of

$$p^2y + 4py + 3y = \sin 2t. \tag{37}$$

Write (37) in the form

$$p^2y + 4py + 3y = 0 + \sin 2t.$$

It is convenient to obtain a solution of (37) consisting of two parts,

$$y = y_1 + u, \tag{38}$$

where $y = y_1$ is the general solution of

$$p^2y + 4py + 3y = 0, \tag{39}$$

and $y = u$ satisfies

$$p^2y + 4py + 3y = \sin 2t. \tag{40}$$

The functions y_1 and u are called respectively the **complementary function** and the **particular integral** of (37). The complementary function contains sufficient arbitrary constants to make the sum $y_1 + u$ the general solution, and the particular integral contains the terms on the right side of the equation which represent applied forces.

The auxiliary equation of (40) is

$$m^2 + 4m + 3 = 0,$$

which has the roots $m = -3, -1$. Thus

$$y_1 = C_1e^{-t} + C_2e^{-3t}. \tag{41}$$

Differentiating (37) twice, there results,

$$p^4y + 4p^3y + 3p^2y = -4 \sin 2t. \tag{42}$$

If (37) is multiplied by 4 and added to (42), the result is

$$p^4y + 4p^3y + 3p^2y + 4(p^2y + 4py + 3y) = 0. \tag{43}$$

This equation is homogeneous. Its auxiliary equation is

$$m^2(m^2 + 4m + 3) + 4(m^2 + 4m + 3) = 0,$$

or

$$(m^2 + 4m + 3)(m^2 + 4) = 0. \tag{44}$$

Attention is called to the fact that if the first factor of (44) is set equal to zero we have the auxiliary equation for Eq. (37). The general solution of (43) is, from (33),

$$y = C_1e^{-t} + C_2e^{-3t} + A \cos 2t + B \sin 2t. \tag{45}$$

Comparison with (41) shows that (45) contains the complementary function. The general solution of (37), since it is of the second order, can contain only two arbitrary constants. To determine the value of two of the constants in (45), this equation is substituted in (37). Since $y = C_1e^{-t} + C_2e^{-3t}$ satisfies (39), it is time saved to substitute only $y = u = A \cos 2t + B \sin 2t$ in (40). Since

$$p^2y = -4A \cos 2t - 4B \sin 2t,$$

$$4py = -8A \sin 2t + 8B \cos 2t,$$

and

$$3y = 3A \cos 2t + 3B \sin 2t,$$

it follows, on substituting these values in (40), that

$$p^2y + 4py + 3y = -(8A + B) \sin 2t + (-A + 8B) \cos 2t = \sin 2t. \tag{46}$$

In order that the equation

$$-(8A + B) \sin 2t + (-A + 8B) \cos 2t - \sin 2t = 0$$

shall be true for all values of t, it is necessary that the coefficients of

sin $2t$ and cos $2t$ each equal zero. Accordingly, equating to zero the coefficients of sin $2t$ and cos $2t$ in (46), there results

$$-8A - B = 1,$$

$$-A + 8B = 0,$$

or

$$A = -\tfrac{8}{65}, \quad B = -\tfrac{1}{65}.$$

Hence the particular integral of (37) is $-\frac{8}{65}\cos 2t - \frac{1}{65}\sin 2t$, and the general solution is

$$y = C_1e^{-t} + C_2e^{-3t} - \tfrac{8}{65}\cos 2t - \tfrac{1}{65}\sin 2t. \tag{47}$$

Example 2. Obtain the general solution of

$$p^2y + 4py + 3y = 4t. \tag{48}$$

The second derivative of Eq. (48) is

$$p^4y + 4p^3y + 3p^2y = 0. \tag{49}$$

The auxiliary equations of (48) and (49) are respectively

$$m^2 + 4m + 3 = 0$$

and

$$m^2(m^2 + 4m + 3) = 0.$$

The general solution of (49) is

$$y = C_1e^{-t} + C_2e^{-3t} + (C_3 + C_4t).$$

Substituting $y = C_3 + C_4t$ in (48), obtain

$$4C_4 + 3(C_3 + C_4t) = 4t,$$

or

$$4C_4 + 3C_3 + (3C_4 - 4)t = 0. \tag{50}$$

By equating to zero the coefficients of each power of t

$$C_3 = -\tfrac{16}{9}, \; C_4 = \tfrac{4}{3}.$$

Hence the particular integral of (48) is $\frac{4}{3}t - \frac{16}{9}$, and the general solution is

$$y = C_1e^{-t} + C_2e^{-3t} + \tfrac{4}{3}t - \tfrac{16}{9}. \tag{51}$$

Example 3. Solve the equation

$$p^2y + 4py + 3y = \sin 2t + 4t. \tag{52}$$

Let the general solution be written

$$y = y_1 + u_1 + u_2, \tag{53}$$

where y_1 satisfies $p^2y + 4py + 3y = 0$,

u_1 satisfies $p^2y + 4py + 3y = \sin 2t$,

u_2 satisfies $p^2y + 4py + 3y = 4t$.

From examples 1 and 2 of this article

$$y_1 = C_1e^{-t} + C_2e^{-3t},$$

$$u_1 = -\tfrac{8}{65}\cos 2t - \tfrac{1}{65}\sin 2t,$$

$$u_2 = \tfrac{4}{3}t - \tfrac{16}{9}.$$

Substituting these values in (53)

$$y = C_1e^{-t} + C_2e^{-3t} - \tfrac{8}{65}\cos 2t - \tfrac{1}{65}\sin 2t + \tfrac{4}{3}t - \tfrac{16}{9}.$$

This function y is a solution of (52) because it satisfies the differential equation. It is the general solution since it contains precisely two arbitrary constants.

If the right side of (52) had contained n terms, each a different elementary function or constant, there would have been n particular integrals $u_1, u_2, \ldots u_n$.

EXAMPLE 4. (*Resonance equation.*) Suppose the mass in the introductory problem of § 6 has a sinusoidal force $F \cos k_1t$ applied to it in the line of its motion. If y denotes the same displacement that was formerly denoted by x, the differential equation of motion becomes

$$p^2y + \frac{k}{M}y = \frac{F}{M}\cos k_1t.$$

The complementary function is by (25)

$$y_1 = A \sin\sqrt{\frac{k}{M}}\,t + B\cos\sqrt{\frac{k}{M}}\,t, \tag{54}$$

which describes the free vibrations of the system. The frequency of the vibrations is $\dfrac{1}{2\pi}\sqrt{\dfrac{k}{M}}$ and is called the **natural frequency** of the system. The natural frequency in this case depends only upon the mass M_1 and the spring coefficient k. The frequency of the applied force is $\dfrac{k_1}{2\pi}$. When this frequency is equal to the natural frequency

the system is said to be in resonance with the applied force.

Suppose resonance exists, that is, $k_1 = \sqrt{\frac{k}{M}}$. The differential equation of motion may then be written

$$p^2y + k_1^2y = \frac{F}{M}\cos k_1t. \tag{55}$$

The complementary function is given by (54). The particular integral of (55) is found by the method of example 1. Differentiating (55) twice

$$p^4y + k_1^2p^2y = -k_1^2\frac{F}{M}\cos k_1t. \tag{56}$$

Combining (55) and (56)

$$p^2(p^2y + k_1^2y) = -k_1^2(p^2y + k_1^2y),$$

or

$$(p^2y + k_1^2y)^2 = 0. \tag{57}$$

The general solution of (57), by Eq. (35), is

$$y = A_1 \sin k_1t + B_1 \cos k_1t + A_2t \sin k_1t + B_2t \cos k_1t. \tag{58}$$

Substituting

$$y = A_2t \sin k_1t + B_2t \cos k_1t$$

in (55) and equating to zero the coefficients of like functions of t, we have

$$A_2 = \frac{F}{2k_1M}, \quad B_2 = 0.$$

If these values of A_2 and B_2 are substituted in (58), the general solution of (55) is seen to be

$$y = A \sin k_1t + B \cos k_1t + \frac{Ft \sin k_1t}{2k_1M}. \tag{59}$$

A More General Equation. We now return to the solution of the equation

$$p^ny + a_1p^{n-1}y + \ldots + a_{n-1}py + a_ny = f(t), \tag{60}$$

where $f(t)$ is a function such that some derivative of $f(t)$ (say the sth) is equal to a constant N times $f(t)$. N may, as in example 2, be zero. Differentiate (60) s times and obtain

$$p^{n+s}y + a_1p^{n+s-1}y + \ldots + a_{n-1}p^{s+1}y + a_np^sy = Nf(t). \tag{61}$$

The elimination of $f(t)$ between (60) and (61) gives

$$p^{n+s}y + \ldots + a_n p^s y - N(p^n y + \ldots + a_n y) = 0. \qquad (62)$$

The characteristic equation of (62) is

$$(m^n + a_1 m^{n-1} + \ldots + a_n)(m^s - M) = 0. \qquad (63)$$

Denote those roots of (63) which are also roots of the characteristic equation of (60) by $m_1, m_2, \ldots m_n$. The general solution of (62), all roots of (63) being distinct, is

$$y = C_1 e^{m_1 t} + \ldots + C_n e^{m_n t} + C_{n+1} e^{m_{n+1} t} + \ldots + C_{n+s} e^{m_{n+s} t}. \qquad (64)$$

The first n terms of the right member of the last equation form the complementary function of (60). To determine the particular integral u substitute

$$y = u = C_{n+1} e^{m_{n+1} t} + \ldots + C_{n+s} e^{m_{n+s} t}$$

in (60) and equate to zero the coefficient of each exponential term. The relations so obtained determine uniquely $C_{n+1}, C_{n+2}, \ldots C_{n+s}$.

If some of the roots of the characteristic equation of (62) are repeated or are complex then the general solution of (62) is modified according to Eqs. (33), (34), or (35). The case of a repeated real root is illustrated in example 2, and that of a repeated complex root in example 4, of this article. Although $f(t)$ here is somewhat limited in nature, it is sufficiently general to include the most frequently occurring cases. However, a method of handling Eq. (60) when $f(t)$ is a general function of applied mathematics is given in § 16.

13. Initial Conditions. Damping. Harmonic Motion. Let us complete the solution of problem (*a*), § (10). Eq. (11) may be written

$$\frac{d^2s}{dt^2} + 3.22\frac{ds}{dt} + 96.6s \quad 0. \qquad (65)$$

The roots of the characteristic equation of (65) are

$$m = -1.61 \pm 9.70i,$$

and the general solution, by Eq. (33), is

$$e^{-1.61t}(A \sin 9.70t + B \cos 9.70t).$$

The last equation may be written, by making use of a trigonometric transformation,

$$s = C_1 e^{-1.61t} \sin (9.70t + \phi), \qquad (66)$$

where C_1 and ϕ are the new arbitrary constants of the general solution. The initial conditions of the problem are

$$\left.\begin{aligned} s &= \tfrac{1}{2} \\ \frac{ds}{dt} &= 0 \end{aligned}\right\} \quad \text{when } t = 0.$$

These conditions give, upon substitution in (66) and the derivative of (66),

$$\tfrac{1}{2} = C_1 \sin \phi,$$

$$0 = C_1(9.70 \cos \phi - 1.61 \sin \phi),$$

or

$$\phi = \tan^{-} \frac{9.70}{1.61} = 80° \; 35', \; C_1 = 0.5068.$$

Thus the equation of motion of the mass M is

$$s = 0.5068e^{-1.61t} \sin (9.70t + 80° \; 35'). \tag{67}$$

In an equation of the form

$$s = Ce^{-at} \sin (\omega t + \phi) \tag{68}$$

the angle ϕ is called the phase angle. (Sometimes **epoch angle.**) The factor e^{-at} is the **damping factor,** and a is the **damping constant.**

Straight-line motion defined by

$$s = C \sin (\omega t + \phi)$$

is called **simple harmonic motion.** The numerical value of C is called the **amplitude** of the motion. The angle ϕ is called the **phase angle,** and $\frac{\omega}{2\pi}$ is the **frequency** of the harmonic motion.

As a further example of the evaluation of the arbitrary constants, suppose that it is desired to find y in Ex. 1, § 12, when the initial conditions are

$$\left.\begin{aligned} y &= 0 \\ py = \frac{dy}{dt} &= 0 \end{aligned}\right\} \quad \text{when } t = 0.$$

The differential equation is

$$p^2y + 4py + 3y = \sin 2t.$$

By (47) its solution is

$$y = C_1e^{-t} + C_2e^{-3t} - \tfrac{8}{65} \cos 2t - \tfrac{1}{65} \sin 2t.$$

Substituting the initial conditions into the solution and its derivative, we obtain

$$0 = C_1 + C_2 - \tfrac{8}{65},$$

$$0 = - C_1 - 3C_2 - \tfrac{2}{65}.$$

From these,

$$C_1 = \tfrac{13}{65}, \qquad C_2 = - \tfrac{5}{65}.$$

Thus

$$y = \tfrac{13}{65}e^{-t} - \tfrac{5}{65}e^{-3t} - \tfrac{8}{65} \cos 2t - \tfrac{1}{65} \sin 2t.$$

It should be noted that the initial conditions are substituted into the general solution and its derivatives, not into only the complementary function and its derivatives.

14. Summary. The steps in solving the homogeneous differential equation (19) are:

(a) Obtain the characteristic equation of Eq. (19) by replacing p by m in the differential equation.

(b) Obtain the roots of the characteristic equation. If this is of higher degree than the third, it may be necessary to employ the methods of Sec. IV, Chap. II.

(c) Write the various terms of the general solution recalling that: To every non-repeated real root m there corresponds a term of the form Ce^{mt}. To every r-fold repeated real root m there corresponds a sum of terms

$$(C_1 + C_2t + \dots + C_rt^{r-1})e^{mt}.$$

To every non-repeated pair of complex roots $a \pm bi$ there corresponds the terms

$$e^{at}(A \sin bt + B \cos bt).$$

To every r-fold repeated pair of complex roots $a \pm bi$ there corresponds the terms

$$e^{at}[(C_1 + C_2t + \dots C_rt^{r-1}) \sin bt + (D_1 + D_2t + \dots + D_rt^{r-1}) \cos bt].$$

The quantities A, B, $C_1, \dots C_r$, $D_1 \dots D_r$ are arbitrary constants.

The steps in solving the non-homogeneous differential equation (36) are:

(a) Obtain the complementary function by the three steps outlined above.

(b) Find the particular integral by the method of § 12 if applicable. If this method fails, employ § 15 or § 16.

EXERCISES AND PROBLEMS

Obtain the solution of the following equations.

1. $p^3y - 7p^2y - py + 7y = 0$.
2. $p^3y + 2p^2y + 2py + y = 0$. $\left(\text{Roots} = -1, \dfrac{-1 \pm \sqrt{3}\,i}{2}.\right)$
3. $p^2y - 2apy + a^2y = 0$.
4. $p^2y - 2apy + (a^2 + k^2)y = 0$.
5. $p^2y - 4y = \sin 5t$.
6. $p^2y - 3py + 2y = e^{3t}$.
7. $p^2y - (2 + k)py + 2ky = e^{kt}$.
8. $(p^2 + k^2)^2y = \sin kt$.
9. Obtain the equation of motion in problem 3, § 10. The initial conditions are

$$\left.\begin{aligned} \theta &= \theta_0 \\ \frac{d\theta}{dt} &= 0 \end{aligned}\right\} \text{when } t = 0.$$

10. Obtain the equation of motion of problem 4, § 10. Give the period and amplitude of the harmonic motion. The weight of a cubic foot of water is 62.4 lb.

11. In the differential equation obtained as the answer to problem 2, § 10, $\sin \theta$ is approximately equal to θ if θ is small. Making this approximation obtain the equation of motion if the initial conditions are

$$\left.\begin{aligned} \theta &= \theta_0 \\ \frac{d\theta}{dt} &= \phi \end{aligned}\right\} \text{when } t = 0.$$

15. Second Method of Solution of Non-homogeneous Linear Differential Equations. Operator Method. The method of § 12 for obtaining the particular integral has the special advantage that it is necessary to remember only a process rather than a number of formulas. On the other hand, it has the disadvantage that it is long. Furthermore, it is applicable only when $p^rf(t) = \text{constant} \times f(t)$. In certain cases of very frequent occurrence, the particular integral is more quickly obtained by an operator (not operational) method.

Define the expression [7]

$$F(p)y \equiv (p^n + a_1p^{n-1} + \ldots + a_{n-1}p + a_n)y$$

by the equation

$$(p^n + a_1p^{n-1} + \ldots + a_n)y \equiv p^ny + a_1p^{n-1}y + \ldots + a_ny. \quad (69)$$

Write Eq. (36)

$$F(p)y = f(t). \quad (70)$$

[7] The symbol $\equiv$ is read "is defined to be."

This equation solved formally for y gives

$$y = \frac{1}{F(p)} f(t). \tag{71}$$

The symbol $\frac{1}{F(p)}$ is thus far meaningless. However, y in (70) is a function of t such that if operated on by $F(p)$ it gives $f(t)$. In other words, $\frac{1}{F(p)} f(t)$ is a particular integral of (70). This indicates a definition for $\frac{1}{F(p)}$. Accordingly, $\frac{1}{F(p)}$ is defined to be an operator which is the inverse of $F(p)$, that is

$$\begin{aligned} F(p)\left[\frac{1}{F(p)} f(t)\right] &= f(t), \\ \frac{1}{F(p)}\,[F(p)\, f(t)] &= f(t). \end{aligned} \tag{72}$$

We make use of (72) in obtaining the particular integral in certain cases.

EXAMPLE 1. Find the particular integral of

$$F(p)y = e^{at}, \quad \text{where } F(a) \neq 0. \tag{73}$$

Now

$$pe^{at} = ae^{at},$$

$$p^n e^{at} = a^n e^{at},$$

and

$$(p^n + a_1 p^{n-1} + \dots + a_n)e^{at} = (a^n + a_1 a^{n-1} + \dots + a_n)e^{at},$$

or

$$F(p)e^{at} = F(a)e^{at}. \tag{74}$$

Applying the operator $\frac{1}{F(p)}$ to Eq. (74) we have, in view of (72),

$$e^{at} = \frac{F(a)e^{at}}{F(p)},$$

or

$$\frac{e^{at}}{F(p)} = \frac{e}{F(a)}. \tag{75}$$

But from (71), $y = \frac{e^{at}}{F(p)}$ is the particular integral desired. Thus the

particular integral of (73) is $y = u \quad \frac{e^{at}}{F(a)}$. In case $F(a) = 0$, the particular integral is obtained by the method of § 12.

For example, let us obtain the particular integral of

$$(p^5 + 6p^4 + 7p^3 + 3p^2 + 11)y = e^{3t}.$$

$$y = u = \frac{e^{3t}}{(3)^5 + 6(3)^4 + 7(3)^3 + 3(3)^2 + 11} = \frac{e^{3t}}{956}.$$

This method of obtaining the particular integral of differential equations like (73) finds an important application in the determination of the steady-state complex number solution of an electric circuit in which an alternating voltage is impressed. Thus a circuit differential equation might be of the form

$$F(p)y = e^{j\omega t},$$

where y is current or charge. The steady-state solution, which is the particular integral, is

$$y = u = \frac{e^{j\omega t}}{F(j\omega)}.$$

A similar application occurs in the case of a mechanical system to which a periodic force is applied.

EXAMPLE 2. Obtain the particular integral of

$$F(p^2)y = E \sin at, \qquad F(-a^2) \neq 0.$$

By a method similar to that employed in example 1 it is easily shown that the particular integral is in this case

$$y = u = \frac{E \sin at}{F(p^2)} = \frac{E \sin at}{F(-a^2)}. \tag{76}$$

If the function of p on the left-hand side of the equation is not a function of p^2, that is involves p as well, as in the example $F(p) = p^2 - 2p + 2$, the procedure then is indicated in the solution of the following example. Obtain the particular integral of

$$(p^2 - 2p + 2)y \qquad E \sin 5t.$$

$$y = u = \frac{E \sin 5t}{p^2 - 2p + 2} = \frac{E \sin 5t}{[p - (1 + i)][p - (1 - i)]}.$$

Multiplying the numerator and denominator of this fraction by

$[p + (1 + i)]\,[p + (1 - i)]$, the denominator becomes a function of p^2. Thus

$$y = u = \frac{(p^2 + 2p + 2)E \sin 5t}{[p^2 - (1 + i)^2]\,[p^2 - (1 - i)^2]}$$

$$= \frac{(p^2 + 2p + 2)E \sin 5t}{[-25 - (1 + i)^2]\,[-25 - (1 - i)^2]}$$

$$= \frac{(p^2 + 2p + 2)}{629} E \sin 5t.$$

And performing the differentiations indicated by p,

$$y = u = \frac{(10 \cos 5t - 23 \sin 5t)E}{629}.$$

In case $F(-a^2) = 0$, recourse is had to the method of § 12, example 4.

If in (76) $\sin at$ is replaced by $\cos at$, a solution may be obtained by a similar method. See § 17.

Example 3. Obtain the particular integral of

$$(p^2 + 2p + 2)y = E \sin 5t + Fe^{3t}.$$

$$y = u = \frac{E \sin 5t}{p^2 + 2p + 2} + \frac{Fe^{3t}}{p^2 + 2p + 2}$$

$$= \frac{E(10 \cos 5t - 23 \sin 5t)}{629} + \frac{Fe^{3t}}{17}$$

(by examples 1 and 2).

Example 4. Methods of proving the following formulas are given in § 123.

$$\frac{1}{F(p)} e^{at} F_1(t) = e^{at} \left[\frac{1}{F(p + a)} F_1(t)\right], \tag{77}$$

$$\frac{1}{F(p)} tF_1(t) = \left[t - \frac{1}{F(p)} F'(p)\right] \frac{1}{F(p)} F_1(t), \tag{78}$$

where $F_1(t)$ is any function of t and $F'(p) = \dfrac{dF(p)}{dp}$.

These relations are used as follows. In formula (77) let

$$a = -1, \quad F_1(t) = E \sin 5t, \quad F(p) = p^2 + 1.$$

Then

$$\frac{1}{p^2+1}\,e^{-t}\,E\sin 5t = Ee^{-t}\left[\frac{\sin 5t}{(p-1)^2+1}\right]$$

$$= Ee^{-t}\left[\frac{\sin 5t}{p^2-2p+2}\right],$$

and by example 2,

$$= \frac{Ee^{-t}\,[10\cos 5t - 23\sin 5t]}{629}.$$

In formula (78) let

$$F(p) = p^2+2p+2, \quad F_1(t) = Ee^t.$$

Then

$$\frac{1}{p^2+2p+2}\,tEe^t = \left[t - \frac{(2p+2)}{p^2+2p+2}\right]\frac{1}{p^2+2p+2}\,Ee^t$$

$$= E\left[t - \frac{(2p+2)}{p^2+2p+2}\right]\frac{e^t}{5}$$

$$= E\left[\frac{te^t}{5} - \frac{(2p+2)e^t}{5(p^2+2p+2)}\right]$$

$$= E\left[\frac{te^t}{5} - \frac{2(p+1)e^t}{25}\right]$$

$$= \frac{Ee^t}{5}\left(t - \frac{4}{5}\right).$$

16. General Method of Solution of Non-homogeneous Linear Differential Equations. A general method is sometimes necessary which is applicable to every $f(t)$ for which $\int_0^t \cdots \int_0^t e^{at}f(t)\,dt^n$ exists. In developing a general method, we first establish the relation

$$(p^n + a_1p^{n-1} + \ldots + a_n)y = (p - m_1)(p - m_2)\ldots(p - m_n)y, \qquad (79)$$

where the left side is defined by (69) and where $m_1, m_2, \ldots, m_n$ are the roots (real or complex) of

$$m^n + a_1m^{n-1} + \ldots + a_{n-1}m + a_n = 0. \qquad (80)$$

If p were merely an algebraic symbol instead of a differential operator, Eq. (79) would follow immediately from the factor theorem of algebra. However, it is not difficult to establish Eq. (79) for $p = \dfrac{d}{dt}$. All

essentials of the proof for the general case are included in the proof of the special case for $n = 2$, that is

$$(p^2 + a_1 p + a_2)y = (p - m_1)(p - m_2)y. \tag{81}$$

That (79) is true for $n = 2$ is easily established as follows. If $n = 1$ in Eq. (69), there results

$$(p - m_1)y = py - m_1 y.$$

If $(p - m_1)y$ be denoted by v, then by (69)

$$(p - m_2)v = pv - m_2 v,$$

or

$$\begin{aligned}(p - m_2)\,[(p - m_1)y] &= p(p - m_1) - m_2(p - m_1)\\ &= p^2 - (m_1 + m_2)p + m_1 m_2.\end{aligned}$$

But the relations between the roots m_1 and m_2 and coefficients a_1 and a_2 of the quadratic equation $p^2 + a_1 p + a_2 = 0$ are

$$-(m_1 + m_2) = a_1, \qquad m_1 m_2 = a_2.$$

Thus

$$(p - m_1)(p - m_2)y = (p^2 + a_1 p + a_2)y.$$

By the same argument, it is evident that

$$(p - m_2)(p - m_1)y = (p^2 + a_1 p + a_2)y,$$

where the order of the factors in the left member is changed.

By obvious extensions of the proof for $n = 2$, the general equation (79) can be proved and by such proof it can be shown that the order of the factors in the right-hand side of (70) is immaterial.

The method of obtaining the general solution of

$$(p - m_1)(p - m_2)\ldots(p - m_n)y = f(t) \tag{82}$$

is clearly explained by consideration of the special case

$$(p - m_1)(p - m_2)y = f(t). \tag{83}$$

Let

$$(p - m_2)y = v. \tag{84}$$

Then (83) is

$$(p - m_1)v = f(t). \tag{85}$$

If Eq. (85) is multiplied by $e^{-m_1 t}$ and both sides of the equation integrated with respect to t, there is obtained

$$e^{-m_1 t}v = \int e^{-m_1 t} f(t)dt + C_1,$$

or

$$= e^{m_1 t}\int e^{-m_1 t} f(t)dt + C_1 e^{m_1 t}. \tag{86}$$

Substituting this value of v in (84), we have the non-homogeneous equation

$$(p - m_2)y = e^{m_1 t}\int e^{-m_1 t} f(t)dt + C_1 e^{m_1 t}. \tag{87}$$

Multiplying (87) by $e^{-m_2 t}$ and integrating both sides with respect to t we obtain

$$e^{-m_2 t}y = \int e^{(m_1 - m_2)t}\left[\int e^{-m_1 t} f(t)dt + C_1\right]dt + C_2,$$

or

$$y = e^{m_2 t}\int e^{(m_1 - m_2)t}\left[\int e^{-m_1 t} f(t)dt + C_1\right]dt + C_2 e^{m_2 t}. \tag{88}$$

Eq. (88) is the general solution of (83). However, it is easier to obtain the complementary function by the method of § 14 and to employ (88) to find only the particular integral. Hence, to obtain the particular integral the arbitrary constants C_1 and C_2, in the solution just obtained, must be set equal to zero.

If Eq. (81) had contained n factors instead of two, a continuation of the process outlined in Eqs. (84–88) would yield either the particular integral or the general solution according as $C_1, \ldots . C_n$ were or were not zero.

This general method holds whether the roots $m_1, m_2, \ldots m_n$ are real or complex.

Example. Obtain the general solution of $(p^2 + 1)y = t$ by the method of § 16.

In $(p - i)(p + i)y = t$, denote $(p + i)y$ by v. We desire first the solution of

$$(p - i)v = t.$$

By Eq. (86)

$$v = it + 1 + Ae^{it},$$

where A is an arbitrary constant. Substituting the value of v just obtained in $(p + i)y = v$, the non-homogeneous equation

$$(p + i)y = it + 1 + Ae^{it}$$

is obtained. Multiplying this equation by e^{it} and integrating both sides with respect to t, we have

$$e^{it}y = te^{it} + \frac{A}{2i}e^{2it} + C_2,$$

or

$$y = \frac{A}{2i}e^{it} + C_2e^{-it} + t$$

$$= C_1e^{it} + C_2e^{-it} + t = C_3 \sin t + C_4 \cos t + t.$$

This is the general solution sought.

In general, the evaluation of the integrals in Eq. (88) will not yield a finite sum of elementary functions. On the other hand, if the integrands are expanded in series and the integrations performed on the series the functional properties of the solution are obscured. Consequently, it is frequently preferable, in an engineering problem, to expand the $f(t)$ of Eq. (82) in a Fourier series (see Sec. III) and obtain the particular integral by the application of $\frac{1}{F(p)}$ to the first few terms of the series as explained in the previous article.

17. Summary. **The general method of § 16 is theoretically applicable to every Eq. (60), where $f(t)$ is any applied force or voltage which is a function of the time. However, if $f(t)$ is not one of the special forms treated in §§ 12 and 15 it is preferable, from an engineering standpoint, to solve the equation by the method described in the last paragraph of § 16.**

If $f(t)$ is one of, or a linear combination of, the forms e^{at}, sin at, cos at, $t\,F_1(t)$, $e^{at}\,F_1(t)$, then the particular integral may be written down at once by means of the following formulas:

$$F(p)' \qquad \frac{e^{at}}{F(a)}, \tag{89}$$

$$F(p^2)\ \sin at \qquad \frac{\sin at}{F(-a^2)}, \tag{90}$$

$$\frac{1}{F(p^2)} \cos at = \frac{\cos at}{F(-a^2)}, \tag{91}$$

$$\frac{1}{F(p)} \cos at = \frac{\phi(p)}{f(-a^2)} \cos at, \tag{92}$$

$$\frac{1}{F(p)} \sin at = \frac{\phi(p)}{f(-a^2)} \sin at, \tag{93}$$

where $\phi(p)$ is a function such that $F(p)\,\phi(p) = f(p^2)$ is a function of p^2.

$$\frac{1}{F(p)}\,e^{at}F_1(t) = e^{at}\left[\frac{1}{F(p+a)}\,F_1(t)\right], \tag{94}$$

$$\frac{1}{F(p)}\,t\,F_1(t) = \left[t - \frac{1}{F(p)}\,F'(p)\right]\frac{1}{F(p)}\,F_1(t). \tag{95}$$

The complementary function in every case is obtained by the method of § 14.

EXERCISES AND PROBLEMS

1. By the method of § 16, obtain the general solution of $(p - a)^2y = 0$.

2. Solve $(p^2 + 1)y = te^{2t}$.

3. Solve $(p^2 + a^2)x = 2D \cos \omega t + 0.5D \cos 2\omega t + E$.

Hint: Let $x = y + k$ and the differential equation becomes $(p^2 + a^2)y = 2D \cos \omega t + 0.5D \cos 2\omega t + E - ka^2$. If $k = \dfrac{E}{a^2}$ the differential equation in y contains no constant term.

4. Solve $(p^4 + 4)y = \sin 3t + \cos 3t$.

5. Solve $\left(p^2 + \dfrac{R}{L}p + \dfrac{1}{LC}\right)y = a_1 \sin t + a_2 \sin 2t + a_3 \sin 3t + b_1 \cos t$
$+ b_2 \cos 2t + b_3 \cos 3t,$
where the only variables are y and t.

The next three exercises are concerned with the solution of three frequently occurring types of differential equations whose coefficients are not necessarily constants.

6. Find the general solution of the first order linear differential equation, $\dfrac{dy}{dt} + Py = Q$, where P and Q are functions of t.

Hint: Multiplying the differential equation by $e^{\int P dt}$ we have

$$e^{\int P dt}\left[\frac{dy}{dt} + Py\right] = e^{\int P dt}Q.$$

Integrating with respect to t

$$e^{\int P dt}y = \int e^{\int P dt}Q\,dt + C.$$

(It is easily verified by differentiation that $e^{\int P dt}y$ is the integral of the left side of the differential equation.) Finally

$$y = e^{-\int P dt}\left[\int e^{\int P dt}Q\,dt + C\right].$$

This is the general solution since it satisfies the differential equation and contains one arbitrary constant.

7. Obtain the general solution of

$$x(1 + y^2)dx + y(1 + x^2)dy = 0.$$

This type is known as **variables separable.** Separating the variables, the equation is

$$\frac{xdx}{1 + x^2} + \frac{ydy}{1 + y^2} = 0.$$

Integrating

$$\tfrac{1}{2} \log (1 + x) + \tfrac{1}{2} \log (1 + y^2) = C_1 = \log C_2$$

or

$$(1 + x^2)(1 + y^2) = C_2^2.$$

8. Obtain the solution of $t^2 \dfrac{d^2y}{dt^2} - t\dfrac{dy}{dt} + 4y = t^2$.

An equation of this form is called a **homogeneous linear differential equation.** By means of the substitution $t = e^x$ it can be reduced to a linear differential equation with constant coefficients. Let $t = e^x$. Then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = e^{-x}\frac{dy}{dx},$$

and

$$\begin{aligned}\frac{d^2y}{dt^2} &= \frac{d}{dt}\left(e^{-x}\frac{dy}{dx}\right)\\ &= \frac{d}{dx}\left(e^{-x}\frac{dy}{dx}\right)\frac{dx}{dt}\\ &= e^{-2x}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right).\end{aligned}$$

Substituting these values of the derivatives in the original equation, it reduces to

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^{2x}.$$

9. Solve the equations:

(a) $3e^x \tan y dx + (1 - e^x) \sec^2 y dy = 0,$

(b) $\dfrac{dy}{dx} = \sqrt{\dfrac{1 - y}{1 - x}},$

(c) $\sin x \cos y dx - \cos x \sin y dy = 0.$

10. Solve the equations:

(a) $\dfrac{dy}{dx} + \dfrac{1}{x}y = \dfrac{1}{x^2},$

(b) $\dfrac{dy}{dx} + xy = x^2,$

(c) $\dfrac{dy}{d\theta} + y \cos \theta = \sin 2\theta.$

11. Solve the equations:

(a) $t^2p^2y + 3tpy + y = 0,$

(b) $t^2p^2y + 2tpy - 6y = 3t^2 + 4,$

(c) $t^2p^2y + 3tpy + y = \sin t.$

18. Simultaneous Linear Differential Equations. Methods of solving simultaneous linear differential equations with constant coefficients are illustrated by the solution of some typical examples.

EXAMPLE 1. Solve the system of differential equations

$$\frac{dx}{dt} + 2x - 3y = 0,$$

$$\frac{dy}{dt} - x + 4y = 0,$$

or

$$(p + 2)x - 3y = 0, \tag{96}$$

$$-x + (p + 4)y = 0. \tag{97}$$

If (97) is multiplied by $(p + 2)$ and added to (96), the equation $(p^2 + 6p + 5)y = 0$ is obtained. The general solution of this equation is

$$y = C_1e^{-t} + C_2e^{-5t}. \tag{98}$$

Substituting this value of y in (97), the value of x is

$$x = 3C_1e^{-t} - C_2e^{-5t}. \tag{99}$$

The system of Eqs. (98–99) is the general solution of the system of differential equations (96–97).

The order of a system of linear differential equations is in general the sum of the orders of the highest derivatives appearing in each of the differential equations. The order of the above system is 2. Consequently, the number of arbitrary constants in (98) and (99) is 2.

If the value of y in Eq. (98) is substituted in (96), the solution of the resulting equation for x gives

$$x = C_3e^{-2t} + 3C_1e^{-t} - C_2e^{-5t}.$$

This value for x and the value of y in (98) will satisfy (97) only in case $C_3 = 0$. Thus, it is immaterial in which of the two differential equations the value of y is substituted; the solution is the same.

EXAMPLE 2. Solve the system of differential equations

$$(p + a_{11})x + a_{12}y = e^{3t} \tag{100}$$

$$a_{21}x + (p + a_{22})y = 0. \tag{101}$$

Multiplying (100) by $-a_{21}$ and (101) by $(p + a_{11})$ and adding the two equations, we have

$$[(p + a_{11})(p + a_{22}) - a_{12}a_{21}]\, y = - a_{21}e^{3t}. \tag{102}$$

Multiplying (100) by $(p + a_{22})$ and (101) by $-a_{12}$ and adding, we have

$$[(p + a_{22})(p + a_{11}) - a_{12}a_{21}]\, x = (3 + a_{22})e^{3t}. \tag{103}$$

Let the distinct roots of $(m + a_{22})(m + a_{11}) - a_{12}a_{21} = 0$ be m_1 and m_2. The general solutions of (102) and (103), respectively, are

$$y = C_1e^{m_1t} + C_2e^{m_2t} - \frac{a_{21}}{Z}\, e^{3t} \tag{104}$$

$$x = C'_1e^{m_1t} + C'_2e^{m_2t} + \frac{(3 + a_{22})e^{3t}}{Z}, \tag{105}$$

where

$$Z = (3 + a_{22})(3 + a_{11}) - a_{12}a_{21}.$$

Since the general solution of the system (100–101) will contain only two arbitrary constants, relations exist between C'_1 and C_1 and between C'_2 and C_2. If (104) and (105) are substituted in either (100) or (101) and the coefficients of e^{m_1t} and e^{m_2t} each set equal to zero, the relations between the constants are found to be

$$C'_1(m_1 + a_{11}) = - a_{12}C_1, \quad C'_2(m_2 + a_{11}) = - a_{12}C_2$$

or their equivalents. If C'_1 and C'_2 are eliminated from (105) by means of the last equations, then (104–105) are the general solution of (100–101), provided $(p + a_{11})(p + a_{22}) - a_{12}a_{21}$ is not equal to zero for every value of p. If $(p + a_{11})(p + a_{22}) - a_{12}a_{21}$ is identically zero, then (100–101) have no solution. Whether a system of simultaneous linear differential equations with constant coefficients has one, none, or an infinitude of solutions depends upon the coefficients of the dependent variables and the non-homogeneous terms or the right-hand side of the differential equations. The reason for this will be made clear in Sec. II.

EXAMPLE 3. Solve Eqs. (17) of § 10, that is

$$(M_1p^2 + k_1 + k_2)s_1 - k_2s_2 = k_1y_0 \sin \frac{\pi Vt}{L}, \tag{106}$$

$$- k_2s_1 + (M_2p^2 + k_2)s_2 = 0. \tag{107}$$

Eliminate s_2 between (106) and (107) and obtain

$$[M_1M_2p^4 + (M_1k_2 + M_2k_1 + M_2k_2)p^2 + k_1k_2]\, s_1$$
$$= k_1y_0(k_2 - M_2a^2) \sin at, \quad (108)$$

where $a = \dfrac{\pi V}{L}$. The roots of the characteristic equation

$$M_1M_2m^4 + (M_1k_2 + M_2k_1 + M_2k_2)m^2 + k_1k_2 = 0 \quad (108$$

are $\pm\, \omega_1 i$, $\pm\, \omega_2 i$, where

$$i\omega_1 = \sqrt{\frac{-(M_1k_2+M_2k_1+M_2k_2)+\sqrt{(M_1k_2+M_2k_1+M_2k_2)^2-4M_1M_2k_1k_2}}{2M_1M_2}}$$

$$i\omega_2 = \sqrt{\frac{-(M_1k_2+M_2k_1+M_2k_2)-\sqrt{(M_1k_2+M_2k_1+M_2k_2)^2-4M_1M_2k_1k_2}}{2M_1M_2}}.$$

If $\omega_1 \neq \omega_2$, then the general solution of (108) is

$$s_1 = C_1 \sin \omega_1 t + C_2 \cos \omega_1 t + C_3 \sin \omega_2 t + C_4 \cos \omega_2 t$$
$$+ k_1 \frac{y_0(k_2 - M_2a^2)}{Z} \sin at, \quad (109)$$

where $\quad Z = M_1M_2a^4 - (M_1k_2 + M_2k_1 + M_2k_2)a^2 + k_1k_2.$

Substituting (109) in (107) and solving the resulting differential equation for s_2, we have

$$s_2 = C_5 \sin \alpha t + C_6 \cos \alpha t$$
$$+ k_2\left(\frac{C_1}{\beta} \sin \omega_1 t + \frac{C_2}{\beta} \cos \omega_1 t + \frac{C_3}{\gamma} \sin \omega_2 t + \frac{C_4}{\gamma} \cos \omega_2 t\right)$$
$$+ \frac{y_0k_1(k_2 - M_2a^2)}{ZZ_1} \sin at, \quad (110)$$

where
$$\alpha = \frac{k_2}{M_2}$$
$$\beta = k_2 - M_2\omega_1^2$$
$$\gamma = k_2 - M_2\omega_2^2$$
$$Z_1 \quad \frac{k_2 - M_2a^2}{k_2}$$

If (110) is substituted in (106) and the coefficients of $\sin at$ and $\cos at$ are set equal to zero, $C_5 = C_6 = 0$. Finally, the general solution of (106–107) is

$$s_1 = C_1 \sin \omega_1 t + C_2 \cos \omega_1 t + C_3 \sin \omega_2 t + C_4 \cos \omega_2 t + F \sin at \qquad (111)$$

$$s_2 = C'_1 \sin \omega_1 t + C'_2 \cos \omega_1 t + C'_3 \sin \omega_2 t + C'_4 \cos \omega_2 t + \frac{F}{Z_1} \sin at, \qquad (112)$$

where $C'_1 = \dfrac{k_2 C_1}{\beta}, \quad C'_2 = \dfrac{k_2 C_2}{\beta}, \quad F = \dfrac{k_1 y_0 (k_2 - M_2 a^2)}{Z},$

$C'_3 = \dfrac{k_2 C_3}{\gamma}, \quad C'_4 = \dfrac{k_2 C_4}{\gamma}.$

Let the spring and tire system represented by Eqs. (106–107) be the wheel and tire represented in Fig. 4c of an automobile such that the constants of the equations are

$M_1 = 3.105$ slugs, $M_2 = 31.05$ slugs, $k_2 = 3000$ lb. per ft.

$k_1 = 13{,}200$ lb. per ft., and $L = 1$ ft.

Let us find the two speeds of the automobile which will cause resonance in the spring and tire system.

If the numerical values for k_1, k_2, M_1, and M_2 are substituted in the formulas for ω_1 and ω_2, the values for ω_1 and ω_2, respectively, are 4.40 and 72.77. Setting $\dfrac{\pi V}{L} = \omega_1$ and $\dfrac{\pi V}{L} = \omega_2$, the two values for V are, respectively, 1.40 and 23.16 ft. per sec. (0.95 and 15.79 miles per hour).

Methods of obtaining the particular integral of the differential equations of mechanical systems having many degrees of freedom or of circuits with many meshes are given in Chap. II, Sec. II. Operational methods of solving such systems of differential equations are given in Chap. IV. Analogies between mechanical and electrical systems are discussed in the paper of Ref. 10 at the end of the text.

19. Electric Circuit Principles. The differential equations for electric circuits with lumped parameters are of exactly the same form as the equations for mechanical systems that were derived and solved in the preceding work. Kirchhoff's electromotive force law plays the same rôle in setting up the former equations as D'Alembert's principle does in setting up the latter. Kirchhoff's laws may be stated:

1. The algebraic sum of the electromotive forces around a closed circuit is zero.
2. The algebraic sum of all the currents into the junction point of a network is zero.

The first law can be shown to be an application of the law of conservation of energy to a circuit; the second, a statement of the conservation of electricity. The algebraic sign of an electromotive force or of a current indicates its direction. It is implied therefore that a positive direction with respect to the current must be specified arbitrarily in order that the symbols representing electromotive force and current may have physical significance.

In addition to electromotive forces applied externally, for example, by batteries and generators, there are electromotive forces due to the current in the circuit elements. If the positive direction for electromotive force is chosen the same as that for current, these are:

$$\text{Electromotive force of self-inductance} = -L\frac{di}{dt},$$

$$\text{Electromotive force of resistance} = -Ri,$$

$$\text{Electromotive force of capacitance} = -\frac{q}{C},$$

where the coefficient of self-inductance L is in henrys, the resistance R in ohms, the capacitance C in farads, the current i in amperes, the charge q in coulombs, the electromotive force e in volts, and time t in seconds. Current and charge are related by the equation

$$i = \frac{dq}{dt}.$$

A comparison with the mechanical principles shows that electric charge is analogous to mechanical displacement, current to velocity, electromotive force to force, inductance to mass, resistance to viscous damping, and reciprocal of capacitance $(1/C)$ to the spring coefficient.

If two circuits are coupled magnetically the electromotive force of mutual inductance in the first due to a change of current in the second is

$$e_1 = -L_{12}\frac{di_2}{dt}$$

and similarly, the electromotive force in the second due to a change of current in the first is

$$e_2 = -L_{21}\frac{di_1}{dt},$$

where L_{12} and L_{21} are positive constants provided that currents flowing in the positive directions in both circuits produce magnetic fluxes that link either circuit in the same direction. If the fluxes due to positive currents link either circuit in the opposite directions the algebraic signs of the electromotive forces of mutual induction are changed. It can be shown that the mutual inductances L_{12} and L_{21} are equal.

20. Derivation of Differential Equations of Simple Linear Circuits. The following differential equations are derived by means of the principles of § 19. The symbols have the same significance as given there.

(*a*) *Simple series circuit.* It is desired to determine the differential equation of the simple series circuit shown in Fig. 7, consisting of an inductance, resistance, and capacitance in series with a battery of constant electromotive force E.

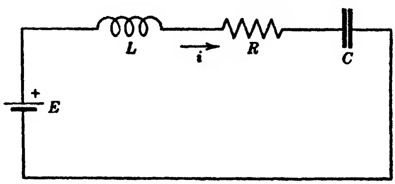


FIG. 7.—Simple Series Circuit.

According to Kirchhoff's first law the algebraic sum of the electromotive forces around the circuit is zero. Let the positive direction for current and voltage be chosen as clockwise. The electromotive forces in that direction are then: $-L\dfrac{di}{dt}$ due to the inductance, $-Ri$ due to the resistance, $-\dfrac{q}{C}$ due to the condenser, and E due to the battery. Thus the equation may be written:

$$-L\frac{di}{dt} - Ri - \frac{q}{C} + E = 0. \tag{113}$$

Or, by using the relation between current and charge on the condenser, (113) may be rewritten, after changing signs throughout,

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} - E \quad 0.$$

(*b*) *Circuits with conductive coupling.* Let us determine the differential equations for the circuit shown in Fig. 8, which contains an alternating voltage $E \sin \omega t$.

According to the first of Kirchhoff's laws the sums of the electromotive forces around the closed circuits $f\ a\ b\ e$ and $b\ c\ d\ e$ must each be zero.

Designate the currents in the three branches by i_1, i_2, and i_3 as shown in the figure, and let the associated arrows indicate the positive direction for both current and electromotive forces in the respective branches. Summing the electromotive forces around the first circuit we obtain

$$-L_1\frac{di_1}{dt} - R_1 i_1 - \frac{q_1}{C_1} - R_{12} i_3 + E \sin \omega t = 0.$$

Around the second circuit

$$-\frac{q_2}{C_2} - R_2 i_2 - L_2 \frac{di_2}{dt} + R_{12} i_3 = 0.$$

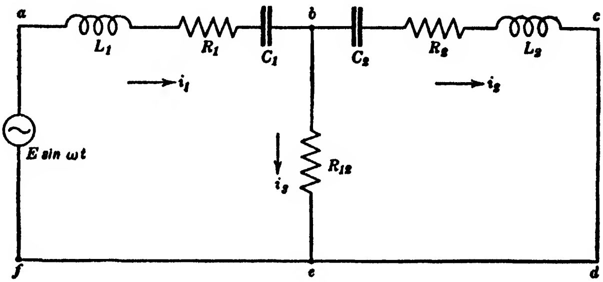


FIG. 8.

By the second of Kirchhoff's laws,

$$i_1 - i_2 - i_3 = 0 \quad \text{or} \quad i_3 = i_1 - i_2.$$

Using the last relation to eliminate i_3 from the first two equations, and rewriting, we obtain

$$L_1\frac{di_1}{dt} + (R_1 + R_{12})i_1 + \frac{q_1}{C_1} - R_{12} i_2 = E \sin \omega t,$$

$$L_2\frac{di_2}{dt} + (R_2 + R_{12})i_2 + \frac{q_2}{C_2} - R_{12} i_1 = 0.$$

Expressing the currents in terms of charges,

$$\left.\begin{aligned} L_1\frac{d^2q_1}{dt^2} + (R_1 + R_{12})\frac{dq_1}{dt} + \frac{q_1}{C_1} - R_{12}\frac{dq_2}{dt} &= E \sin \omega t, \\ L_2\frac{d^2q_2}{dt^2} + (R_1 + R_{12})\frac{dq_2}{dt} + \frac{q_2}{C_2} - R_{12}\frac{dq_1}{dt} &= 0. \end{aligned}\right\} \qquad (114)$$

(*c*) *Circuits with condensive coupling.* Let us obtain the differential equations for the currents in the condensively coupled circuits of

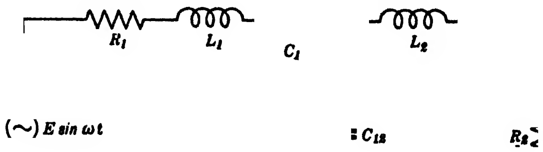


FIG. 9.

Fig. 9. By the same procedure as used in the previous example the following equations are found:

$$
\begin{aligned}
L_1 \frac{d^2q_1}{dt^2} + R_1 \frac{dq_1}{dt} + \left(\frac{1}{C_1} + \frac{1}{C_{12}}\right)q_1 - \frac{q_2}{C_{12}} &= E \sin \omega t, \\
L_2 \frac{d^2q_2}{dt^2} + R_2 \frac{dq_2}{dt} + \frac{q_2}{C_{12}} - \frac{q_1}{C_{12}} &= 0.
\end{aligned}
\tag{115}
$$

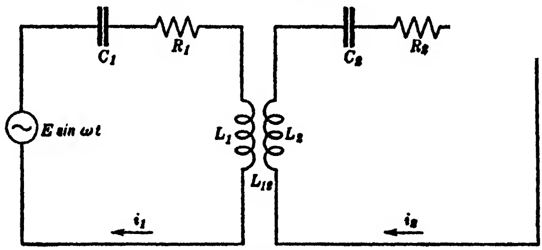


FIG. 10.

(*d*) *Circuits with transformer coupling.* Let us obtain the differential equations for the currents in the inductively coupled circuits of Fig. 10.

By the first of Kirchhoff's laws the sum of the electromotive forces around each circuit is zero. Let the positive direction for current and voltage be the same in each circuit, and let the positive directions for the two circuits be so related that positive currents cause fluxes in the same direction in the magnetic circuit. In the first circuit the electromotive forces include: $E \sin \omega t$ due to the generator, $-\frac{q_1}{C_1}$ due to the condenser, $-R_1i_1$ due to the resistance, $-L_1 \frac{di_1}{dt}$ due to self-

induction, and $-L_{12}\dfrac{di_2}{dt}$ due to mutual inductance. Thus the equation of the first circuit becomes

$$L_1\frac{di_1}{dt} + R_1 i_1 + \frac{q_1}{C_1} + L_{12}\frac{di_2}{dt} = E\sin\omega t.$$

Similarly, that of the second is

$$L_2\frac{di_2}{dt} + R_2 i_2 + \frac{q_2}{C_2} + L_{12}\frac{di_1}{dt} = 0.$$

Using the relation between currents and charges, we obtain

$$\begin{aligned} L_1\frac{d^2q_1}{dt^2} + R_1\frac{dq_1}{dt} + \frac{q_1}{C_1} + L_{12}\frac{d^2q_2}{dt^2} &= E\sin\omega t, \\ L_2\frac{d^2q_2}{dt^2} + R_2\frac{dq_2}{dt} + \frac{q_2}{C_2} + L_{12}\frac{d^2q_1}{dt^2} &= 0. \end{aligned} \tag{116}$$

For the set-up of differential equations of motion of complicated electrical, mechanical, or electro-mechanical systems see Ref. 7 at end of text.

PROBLEMS

1. The differential equations obtained in problem 5, § 10, are

$$M_2\frac{d^2s_2}{dt^2} + k_d\left(\frac{ds_2}{dt} - \frac{ds_1}{dt}\right) + k_2(s_2 - s_1) = 0,$$

$$M_1\frac{d^2s_1}{dt^2} - k_d\left(\frac{ds_2}{dt} - \frac{ds_1}{dt}\right) + k_1s_1 - k_2(s_2 - s_1) = 0.$$

Obtain the general solution of these simultaneous equations.

2. The differential equations obtained in problem 6, §10, are

$$a(m_1 + m_2)\frac{d^2\theta_1}{dt^2} + bm_2\frac{d^2\theta_2}{dt^2} + (m_1 + m_2)g\theta_1 = 0,$$

$$a\frac{d^2\theta_1}{dt^2} + b\frac{d^2\theta_2}{dt^2} + g\theta_2 = 0.$$

Obtain the general solution of these simultaneous equations.

3. Find the charge on the condenser in terms of time t in the network represented in Fig. 11 if the current through inductance and all charges are zero at time $t = 0$.

4. Find the currents in each branch and the charge on the condensers in terms of

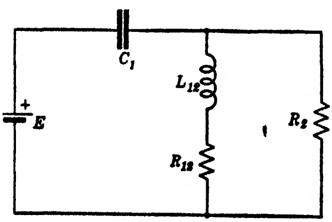


FIG. 11.

time in the network represented in Fig. 12 if all currents and charges are zero at time $t = 0$.

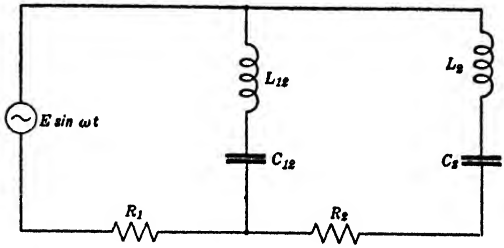


FIG. 12.

5. Write the differential equations for the circuit of Fig. 13.

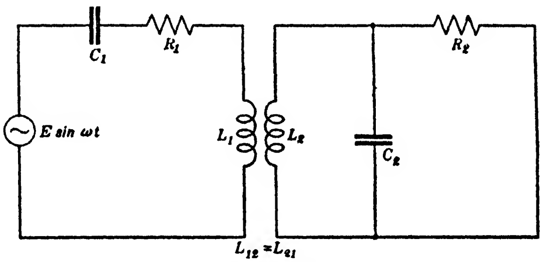


FIG. 13.

6. Obtain an equivalent electrical circuit for the differential equations of problem 1.

7. A uniform beam of weight W and length l is hinged at B and supported in a horizontal position at A by a spring as shown in Fig. 14. The spring constant is k.

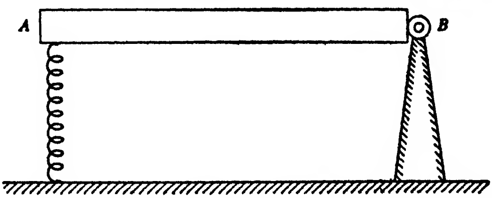


FIG. 14.

The left end of the beam is depressed slightly and suddenly released. Assuming that the beam is rigid, find the differential equation of motion and the period of oscillation.

8. A uniform beam of mass M and length $2l$ is supported on two springs s_1 and s_2, as shown in Fig. 15, and such that the beam has but two degrees of freedom; one an oscillation of the center of gravity in a vertical line, and the other a rotation about a line through the center of gravity and perpendicular to the plane of the figure.

Find the equations of motion and the periods of oscillation for free vibrations of the system.

9. An inextensible string is coiled around a rough circular homogeneous cylinder of mass M and radius r. One end of the string is attached to a stationary point of a horizontal plane such that when the cylinder is rolled up the string and touching the

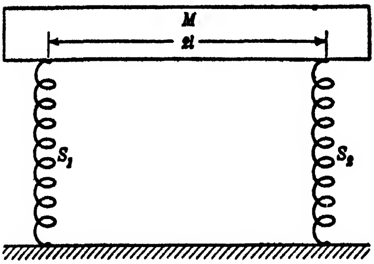


FIG. 15.

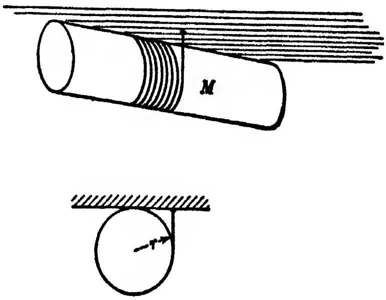


FIG. 16.

plane the string is vertical at its first point of tangency to the cylinder as shown in Fig. 16. The cylinder is dropped. Write the differential equations of motion assuming that the axis of the cylinder is constrained so that it remains horizontal.

10. A body of mass M starts from rest on the rim of a hemispherical bowl of radius r and slides down the inside of the bowl under the influence of gravity. The friction force acts tangentially to the surface and is proportional to the normal force between the weight and the surface. Write the differential equation of the motion of the body, assuming that its dimensions are small compared to the radius r.

11. A pulley has a radius of 1 ft., radius of gyration 6 in., and weight 200 lb. A rope passing over the pulley is attached to a weight of 90 lb. on one side and to a spring on the other. The constant of the spring is 10 lb. per in. of deflection. The system is initially at rest with a 2-in. deflection in the spring, and is then allowed to move under gravity. Obtain the equation of motion and the period of oscillation.

12. Some passenger elevators have been equipped with an air-cushion safety device intended to bring the elevator to a safe stop in case of a free fall. The car is made to fit the shaft closely, thus acting as a piston in a cylinder. This close fit exists only near the bottom of the shaft.

An elevator 5 ft. square weighing 3 tons was traveling upward at the rate of 800 ft. per min. When it reached a height of 700 ft. above the ground floor, the cables broke. The car came to a stop and then fell freely to a point 100 ft. above the ground floor, where the air-cushion safety device began. Assuming adiabatic [8] compression of the air, find the position of the car at any time t.

[8] When air is adiabatically compressed the pressure and volume are related by the equation $PV^{1.4} = \text{constant}$.

II

DETERMINANTS

The following are a few of the numerous engineering applications of determinants. Determinants are advantageously employed in solving linear homogeneous and non-homogeneous algebraic equations, and in giving a criterion for independence of linear algebraic equations. Linear homogeneous and non-homogeneous algebraic equations may arise, for example, in the solution of simultaneous differential equations with constant coefficients. The proof of Bromwich's fundamental theorem of the operational calculus makes extensive use of determinants. The criterion for stability of electrical and mechanical systems, whose differential equations are linear with constant coefficients, is most conveniently expressed in determinant form. The applications of dyadics in synchronous-machine theory and in the theory of elasticity are frequently made in determinant form. The study of equivalent circuits is facilitated by use of determinants. This section is concerned with a brief introduction to some of the important properties and theorems of determinants and with some of the above applications. Most of the applications, however, occur later in the text.

21. Introductory Problem. Before considering the properties of determinants, let us see in a simple example how they may be used in the solution of algebraic equations. Suppose that it is desired to solve the two linear equations

$$\left.\begin{aligned} a_1x + b_1y &= k_1, \\ a_2x + b_2y &= k_2. \end{aligned}\right\} \qquad (117)$$

If we multiply the first equation of (117) by b_2 and the second by $-b_1$ and add, we have

$$(a_1b_2 - a_2b_1)x = k_1b_2 - k_2b_1. \qquad (118)$$

The binomial expression $(a_1b_2 - a_2b_1)$ may be represented by the symbol

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}. \qquad (119)$$

Later the binomial will be seen, by definition, to be a determinant of second order. (Sometimes the symbol itself is called a determinant.)

Using a symbol similar to (119) to designate $(k_1b_2 - k_2b_1)$, we may write as the solution of Eq. (118),

$$x = \frac{(k_1b_2 - k_2b_1)}{(a_1b_2 - a_2b_1)} = \frac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \tag{120}$$

Similarly, multiplying Eqs. (117) by $-a_2$ and a_1 and adding, the solution for y becomes

$$y = \frac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \tag{121}$$

Eqs. (120) and (121) express in symbols the solution of Eqs. (117) as the quotients of two binomials or determinants. The solution of a system of n non-homogeneous equations in n unknowns may be expressed just as simply in the symbolism. If n is greater than 2, the method ordinarily employed in solving the equations becomes laborious, and the use of determinants may effect a saving of time and labor. The procedure is essentially to express the solution as the quotients of determinants, and then to evaluate them by algebraic operations. This leads us (from the engineering viewpoint) to the study of the relations between the determinant, which is a polynomial, and its symbol, which is a square array of letters or numbers called elements of the determinant. We desire to know how to expand the symbol into the polynomial it represents, how to simplify it to make the expansion easier, and the effect upon the value of the determinant of certain operations on its symbol. Lack of space prevents the inclusion of the proofs of most of the theorems given.

22. Definitions. Determinants of the fourth order and of the nth order are denoted respectively by the symbols:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{n1} & \cdots\cdots & \cdots & a_{nn} \end{vmatrix}.$$

Abbreviations for these two symbols are respectively $|\,a_{44}\,|$ and $|\,a_{nn}\,|$.

An nth order determinant is a certain homogeneous polynomial[9] of the nth degree in the n^2 elements a_{ij}, where i and j represent integers from 1 to n inclusive. The explicit form of this polynomial is given in the next paragraph. The elements may be constants or variables, and in the general case when they are not given numerically, each element is designated by a double subscript which indicates the row and column in which the element may be found. For example, a_{42} is the element from the fourth row and the second column. The symbol of the nth order determinant is composed of n horizontal rows and n vertical columns of the elements. A determinant of order $n - 1$ may be formed from a determinant of order n by striking out or erasing any row, say the ith, and any column, say the jth, intersecting at the element a_{ij}. Such a determinant of order $n - 1$ is called the minor (strictly speaking, the first minor) of a_{ij}. This minor is denoted by M_{ij}, where the subscripts are the same as those of the element common to the struck-out column and row. For example, in the fourth-order determinant given above, the minor of a_{23} is

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}.$$

The explicit form of the polynomials referred to in the preceding paragraph is as follows:

$$| a_{22} | \equiv \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \equiv a_{11}a_{22} - a_{12}a_{21}.$$

Similarly,

$$| a_{33} | \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \equiv \begin{array}{r} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} \\ + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} \\ - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \end{array}$$

That is, the polynomial is the sum of all the different products that can be formed from the symbol by taking one element from each row and one element from each column, the sign of the product depending upon the way elements are chosen. For a fourth-order determinant

$$| a_{44} | = \Sigma \pm a_{1q}a_{2r}a_{3s}a_{4t},$$

where q, r, s, t is any one of the 24 permutations of 1, 2, 3, 4. The sign of each term is $+$ or $-$ according as an even or odd number of inter-

[9] A homogeneous polynomial is one all of whose terms are of the same order in the variables; e.g., $x^4 + 13x^2y^2 - 4xy^3 + yx^3 + y^4$ is homogeneous of the fourth order in x and y.

changes is necessary to derive the arrangement q, r, s, t from 1, 2, 3, 4. To illustrate

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Here $a_{11}a_{22}$ has the plus sign since the order of the second subscripts is 1, 2. The second term has the minus sign since one interchange is necessary to bring 2, 1 into the order 1, 2.

Or, in the above example of the third-order determinant, we have for each term of the right member the order of the second subscripts given in the following table. The number of interchanges necessary to put it in the order 123 is given; if this number is even, the sign of the term is positive, if odd it is negative.

Term	Order	No. of Interchanges to Put in Order 123	Sign
First term..................	123	0	+
Second term.................	231	2	+
Third term..................	312	2	+
Fourth term.................	132	1	−
Fifth term..................	213	1	−
Sixth term..................	321	3	−

Finally, the polynomial form of the nth order determinant is

$$|a_{nn}| = \Sigma\,(-1)^i a_{1i_1}a_{2i_2} \ldots . \, a_{ni_n},$$

where $i_1, i_2, i_3, \ldots i_n$ is an arrangement of $1, 2, \ldots n$ derived from $1, 2, \ldots n$ by i interchanges.

The definition of a determinant has no direct application in engineering, but theorems for the evaluation and manipulation of determinants are proved directly from it.

23. Laplace's Expansion. Laplace's expansion is a convenient method of finding the polynomial corresponding to a given symbol. The rule for Laplace's expansion of a determinant of the nth order is:

(a) Form the n products $a_{ij}M_{ij}$, where either i or j is fixed while the other takes the values 1 to n. This corresponds to finding all the products, along any one column or row, of each element by its minor.

As an example let us expand the third-order determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

along the second row. Then $i = 2, j = 1, 2, 3$. The three products are:

$$a_{21}M_{21}, \quad a_{22}M_{22}, \quad \text{and} \quad a_{23}M_{23}.$$

(*b*) Attach to each product its sign as determined from the checkerboard array where the sign attached to $a_{11}M_{11}$ is always plus. This array of signs corresponds to the array of elements in the symbol (third- and fourth-order determinants):

$$\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array} \qquad\qquad \begin{array}{cccc} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{array}$$

(or give $a_{ij}M_{ij}$ the sign $(-1)^{i+j}$, which is the same thing).

The signs of the above products thus are $-a_{21}M_{21}$, $+a_{22}M_{22}$, $-a_{23}M_{23}$.

(*c*) Form the algebraic sum of the n products. We have now expressed the nth order determinant as the algebraic sum of n determinants of order $n - 1$.

The algebraic sum of the products in the example is

$$\Delta = -a_{21}M_{21} + a_{22}M_{22} - a_{23}M_{23}.$$

(*d*) Continue to apply steps (*a*), (*b*), and (*c*) to the second, third, etc., minors until M_{ij} is of order one.

Expanding M_{21} by its first row

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \qquad a_{12}a_{33} - a_{13}a_{32}.$$

Similarly

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \qquad a_{11}a_{33} - a_{13}a_{31},$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \qquad a_{11}a_{32} - a_{12}a_{31}.$$

Thus

$$\begin{aligned} \Delta = &- a_{21}a_{12}a_{33} + a_{21}a_{13}a_{32} + a_{22}a_{11}a_{33} \\ &- a_{22}a_{13}a_{31} - a_{23}a_{11}a_{32} + a_{23}a_{12}a_{31}, \end{aligned}$$

which is equal to the previous definition for the third-order determinant although the arrangement of terms and factors is not the same.

The proof of Laplace's expansion for a determinant of the third order may be completed by making the other five possible expansions

(three rows and three columns or six expansions in all) by Laplace's rule and comparing each result with the definition of a determinant. For a proof of Laplace's expansion for nth-order determinants, see Ref. 11 at the end of the text.

24. Theorems Regarding the Expansion of Determinants. By means of the definition of a determinant and the Laplacian expansion, the following theorems are easily proved. In work with numerical determinants, the value of the following theorems cannot be overemphasized.

(*a*) A determinant is changed in sign by the interchange of any two of its columns (or rows). For example,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{13} & a_{12} & a_{14} \\ a_{21} & a_{23} & a_{22} & a_{24} \\ a_{31} & a_{33} & a_{32} & a_{34} \\ a_{41} & a_{43} & a_{42} & a_{44} \end{vmatrix}.$$

(*b*) A determinant is zero if any two of its rows or any two of its columns are alike. For example,

$$\begin{vmatrix} 3 & 2 & 3 & 4 \\ 1 & 5 & 1 & 0 \\ 7 & 1 & 7 & 9 \\ 8 & 1 & 8 & 2 \end{vmatrix} = 0.$$

(*c*) A determinant can be expanded (in Laplace's expansion) by the elements of any row or any column.

(*d*) The value of a determinant is not altered if the rows be written as columns, and the columns as rows. For example,

$$\begin{vmatrix} 3 & 4 & 4 & 1 \\ 2 & 1 & 2 & 2 \\ 0 & 3 & 5 & 4 \\ 9 & 1 & 7 & 6 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 0 & 9 \\ 4 & 1 & 3 & 1 \\ 4 & 2 & 5 & 7 \\ 1 & 2 & 4 & 6 \end{vmatrix}.$$

(*e*) A common factor of all the elements of any row or column of a determinant may be divided out of the elements and placed as a factor before the new determinant. For example,

$$\begin{vmatrix} 1 & 3 & 2 & 2 \\ 2 & 8 & 6 & 4 \\ 9 & 0 & 1 & 1 \\ 7 & 2 & 2 & 0 \end{vmatrix} = 2 \begin{vmatrix} 1 & 3 & 2 & 2 \\ 1 & 4 & 3 & 2 \\ 9 & 0 & 1 & 1 \\ 7 & 2 & 2 & 0 \end{vmatrix}.$$

(*f*) A determinant is not changed in value if we add to the elements of any row (column) the products of the corresponding elements of another row (column) by the same number. For example, multiplying the third column by 4 and adding to the second column

$$\begin{vmatrix} 2 & 1 & 5 & 3 \\ 3 & 4 & 0 & 2 \\ 1 & 4 & 0 & 9 \\ 1 & 3 & 8 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 1+4\times 5 & 5 & 3 \\ 3 & 4+0 & 0 & 2 \\ 1 & 4+0 & 0 & 9 \\ 1 & 3+4\times 8 & 8 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 21 & 5 & 3 \\ 3 & 4 & 0 & 2 \\ 1 & 4 & 0 & 9 \\ 1 & 35 & 8 & 6 \end{vmatrix}.$$

By means of this theorem it is frequently possible to transform a determinant to an equivalent one in which all elements but one of a row (column) are zero, and thus simplify the expansion by Laplace's rule, as in example 1 below. The following examples further illustrate these theorems.

Example 1. Evaluate the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}.$$

If the first column is multiplied by -1 and the result added successively to the second, third, and fourth columns of the determinant, we have by theorem (*f*)

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 5 & 9 \\ 1 & 3 & 9 & 19 \end{vmatrix}.$$

Applying Laplace's rule to the first row we have

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 9 \\ 3 & 9 & 19 \end{vmatrix}.$$

If the first column is first multiplied by -2 and added to the second and then multiplied by -3 and added to the third column, the determinant is

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 3 & 3 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 3 & 10 \end{vmatrix} = 1.$$

Example 2. Evaluate the determinant

$$\Delta = \begin{vmatrix} 7 & 3 & 2 \\ 14 & 9 & 4 \\ 21 & 27 & 8 \end{vmatrix}$$

By theorem (*e*)

$$7.3.2 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 9 & 4 \end{vmatrix}$$

By theorem (*f*), subtracting the first column from the second and third columns,

$$\Delta = 7.3.2 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 6 & 1 \end{vmatrix} = 42.$$

Example 3. Evaluate the determinant

$$\Delta = \begin{vmatrix} 1 & 7 & 7 \\ 2 & 14 & 3 \\ 3 & 21 & 1 \end{vmatrix}.$$

By theorems (*e*) and (*b*)

$$\Delta = 7 \begin{vmatrix} 1 & 1 & 7 \\ 2 & 2 & 3 \\ 3 & 3 & 1 \end{vmatrix} = 0.$$

Example 4. Prove that

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (y - z)(z - x)(x - y).$$

The factor theorem of elementary algebra states that, if $a_0x^n + a_1x^{n-1} + \ldots + a_n$ vanishes for $x = a$, then $x - a$ is a factor of $a_0x^n + a_1x^{n-1} + \ldots + a_n$. The determinant Δ is a polynomial in x. (Also in y and z.) By theorem (*b*), $\Delta = 0$ for $x = y$. Hence by the factor theorem $x - y$ is a factor of Δ. By the same reasoning, $z - x$ and $y - z$ are factors. To see that there can be no additional factors, it is easy to compare the terms of the expanded product with the expanded determinant written as a polynomial.

25. Multiplication of Determinants. We multiply two third-order determinants. For brevity n in this case is 3, but the method applies for any n.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} =$$

$$\begin{array}{lll} a_{11}\,b_{11} + a_{12}\,b_{21} + a_{13}\,b_{31} & a_{11}\,b_{12} + a_{12}\,b_{22} + a_{13}\,b_{32} & a_{11}\,b_{13} + a_{12}\,b_{23} + a_{13}\,b_{33} \\ a_{21}\,b_{11} + a_{22}\,b_{21} + a_{23}\,b_{31} & a_{21}\,b_{12} + a_{22}\,b_{22} + a_{23}\,b_{32} & a_{21}\,b_{13} + a_{22}\,b_{23} + a_{23}\,b_{33} \\ a_{31}\,b_{11} + a_{32}\,b_{21} + a_{33}\,b_{31} & a_{31}\,b_{12} + a_{32}\,b_{22} + a_{33}\,b_{32} & a_{31}\,b_{13} + a_{32}\,b_{23} + a_{33}\,b_{33} \end{array}$$

Or, in general, denoting an element of the product by $(ab)_{ij}$, for an nth-order determinant

$$(ab)_{ij} = \sum_{k=1}^{k=n} a_{ik}b_{kj}.$$

Right-hand and left-hand multiplications are equivalent, that is

$$b_{nn} \,|\, = \,|\, b_{nn} \qquad |\, a_{nn} \,|$$

26. Application of Determinants to Non-Homogeneous [10] Linear Equations. In § 21, determinants were related to the solution of two equations in two unknowns. We state, but do not prove, Cramer's rule for solving a system of n equations. If in the equations

$$\begin{array}{l} a_{11}x_1 + \ldots + a_{1n}x_n = k_1, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ a_{n1}x_1 + \ldots + a_{nn}x_n = k_n, \end{array} \tag{122}$$

the determinant

$$\Delta = \begin{vmatrix} a_{11} & \ldots\ldots & a_{1n} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ a_{n1} & \ldots\ldots & a_{nn} \end{vmatrix} \neq 0,$$

the equations have exactly one solution, namely,

$$x_1 = \frac{\begin{vmatrix} k_1\, a_{12} \ldots a_{1n} \\ k_2\, a_{22} \ldots a_{2n} \\ \cdot \\ \cdot \\ \cdot \\ k_n\, a_{n2} \ldots a_{nn} \end{vmatrix}}{\Delta}, \quad x_2 = \frac{\begin{vmatrix} a_{11}\, k_1\, a_{13} \ldots a_{1n} \\ a_{21}\, k_2\, a_{23} \ldots a_{2n} \\ \cdot \\ \cdot \\ \cdot \\ a_{n1}\, k_n\, a_{n3} \ldots a_{nn} \end{vmatrix}}{\Delta}, \ldots, x_n = \frac{\begin{vmatrix} a_{11}\, a_{12} \ldots a_{1n-1} k_1 \\ a_{21}\, a_{22} \ldots a_{2n-1} k_2 \\ \cdot \\ \cdot \\ \cdot \\ a_{n1}\, a_{n2} \ldots a_{nn-1} k_n \end{vmatrix}}{\Delta}. \tag{123}$$

[10] A homogeneous equation is one all of whose terms are of the same degree in the variables. For example, if we consider linear equations, $5x + y + 6z = 0$ is a homogeneous equation; but $5x + y + 6z = 4$ is non-homogeneous because of the term on the right side.

The determinants which are the numerators of the fractions giving the values of $x_1, x_2, \ldots x_n$ will be hereafter denoted respectively by the letters $D_1, D_2, \ldots D_n$, the denominator by Δ.

If the determinant Δ of the coefficients of the unknowns in Eqs. (122) is zero, the investigation of solutions is more complicated. In this case, it is advantageous to employ the notion of the rank of a determinant. If a determinant of the nth order is not zero, it is said to be of order n. If the determinant is zero and also every $(r + 1)$-rowed minor formed from it is zero while there is at least one r-rowed minor which is not zero, then the determinant is said to be of **rank** r. For example, if a determinant of nth order is zero, but not all its minors of order $n - 1$ are zero; the determinant is of rank $n - 1$. If all the minors of order $n - 1$ are zero but there is at least one minor of order $n - 2$ which is not zero the nth-ordered determinant is of rank $n - 2$. By means of the idea of rank the facts regarding the solution of n non-homogeneous linear equations in n unknowns are stated and illustrated as follows.

(*a*) If the determinant Δ of the coefficients of the unknowns in Eqs. (122) is not zero, there exists a unique solution which is given by Cramer's rule, i.e., Eqs. (123).

(*b*) If Δ is of rank $r < n$ and any of the determinants $D_1, D_2, \ldots D_n$ of Eqs. (123) are of rank greater than r, there is no solution of the system of Eqs. (122).

Example. Discuss the solution of the system of equations:

$$x + y + z = 1,$$

$$2x + 4y + 2z = 4,$$

$$x - 3y + z = 2.$$

In this system, the rank of Δ is 2. (The first and third columns being equal, $\Delta = 0$, but 2-rowed minors which are not zero can be formed from Δ). The rank of at least one of the three determinants D_1, D_2, D_3 is 3. Consequently, no solution of the system exists.

(*c*) If Δ is of rank $r < n$ and the rank of $D_1, D_2, \ldots D_n$ does not exceed r, then there exist infinitely many sets of solutions of the system (122). The method of obtaining these sets is as follows:

Since Δ is of rank r, the equations and variables in the equations can be so arranged that the upper left-hand r-rowed minor of Δ will be of rank r. Consider the first r equations of the rearranged system. Assign arbitrary values (say $x'_{r+1}, x'_{r+2} \ldots x'_n$) to the last $n - r$ vari-

ables of these r equations, and transpose the results to the right-hand side of the equations. The system of the first r equations then is

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1r}x_r &= k_1 - a_{1r+1}x'_{r+1} \ldots - a_{1n}x'_n, \\ & \\ a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rr}x_r &= k_r - a_{rr+1}x'_{r+1} \ldots - a_{rn}x'_n. \end{aligned}\right\} \tag{124}$$

Eqs. (124) may be solved for $x_1, \ldots x_r$ by Cramer's rule. It is then true that the values obtained for $x_1, \ldots x_r$ along with the values $x'_{r+1}, \ldots x'_n$ will satisfy the remaining $n - r$ equations of the n equations in n unknowns. But $x'_{r+1}, \ldots x'_n$ are arbitrary, and consequently there exist infinitely many sets of solutions.

Example. Obtain sets of solutions of

$$\begin{aligned} 3x + 4y - z - 6w &= 1, \\ 4x + 8y - 2z - 8w &= 2, \\ 5x + 4y - z - 10w &= 1, \\ 3x + 8y - 2z - 6w &= 2. \end{aligned}$$

It can be shown that Δ, D_1, D_2, D_3, and D_4 are each of rank 2. Since the minor $M = \begin{vmatrix} 3 & 4 \\ 4 & 8 \end{vmatrix}$ is of rank 2 (that is, $M \neq 0$), it is not necessary to rearrange either the equations or the unknowns. Assign the arbitrary values z_1 and w_1, respectively, to z and w, and write

$$\begin{aligned} 3x + 4y &= 1 + z_1 + 6w_1, \\ 4x + 8y &= 2 + 2z_1 + 8w_1. \end{aligned}$$

Solving for x and y, we have

$$x = 2w_1,$$

$$y = \frac{1 + z_1}{4}.$$

By substitution in the last two equations of the given system, $x = 2w_1$, $y = \dfrac{1 + z_1}{4}$, $z = z_1$, $w = w_1$ is seen to be a solution of the system.

27. Application of Determinants to Homogeneous Linear Equations. Consider the set of n homogeneous linear equations in n unknowns

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0, \\ & \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= 0, \end{aligned}\right. \tag{125}$$

Eqs. (125) have the trivial solution $x_1 = x_2 = \ldots = x_n = 0$. A necessary and sufficient condition that (125) have a solution other than the trivial one $x_1 = x_2 = \ldots = x_n = 0$ is that the determinant Δ of the coefficients vanish. Or in other words, if the system (125) has a solution other than $x_1 = x_2 = \ldots = x_n = 0$ then $\Delta = 0$, and if $\Delta = 0$ then the system has a solution other than $x_1 = x_2 = \ldots = x_n = 0$.

The non-trivial solutions of (125) are found in much the same way that the solutions of the non-homogeneous equations (122) were found under case (*c*) of § 26. Suppose that Δ is of rank $r < n$. Then Eqs. (125) and the unknowns in the equations can be arranged so that an r-rowed minor of rank r appears in the upper left-hand corner of Δ. Consider the first r equations of the rearranged system. Assign arbitrary values (say $x'_{r+1}, \ldots, x'_n$) to the last $n - r$ variables of these r equations and transpose these terms to the right-hand side of the equations. We then have Eqs. (124) where $k_1 = k_2 = \ldots = k_n = 0$. These equations can be solved by Cramer's rule for $x_1, \ldots, x_r$. It is then true that the values obtained for $x_1, \ldots, x_r$ along with the arbitrary values $x'_{r+1}, \ldots, x'_n$ will satisfy the remaining $n - r$ equations of the n equations in n unknowns. Since $x'_{r+1}, \ldots, x'_n$ are arbitrary, there exist infinitely many sets of solutions.

Example. Obtain sets of solutions of

$$3x + 4y - z - 6w = 0,$$

$$4x + 8y - 2z - 8w = 0,$$

$$5x + 4y - z - 10w = 0,$$

$$3x + 8y - 2z - 6w = 0,$$

in which Δ is the same as in the previous example. Evidently Δ, D_1, D_2, D_3, and D_4 are of rank 2. Since the minor $M = \begin{vmatrix} 3 & 4 \\ 4 & 8 \end{vmatrix}$ is of rank 2, it is not necessary to rearrange either the equations or the unknowns in the equations.

We write

$$3x + 4y = z_1 + 6w_1,$$

$$4x + 8y = 2z_1 + 8w_1.$$

Hence

$$x = 2w_1,$$

$$y = \frac{z_1}{4}.$$

The values $x = 2w_1$, $y = \frac{z_1}{4}$, $z = z_1$, $w = w_1$ satisfy the system of equations whose solution is desired. Since z_1 and w_1 are arbitrary, we have obtained sets of solutions. The case of m homogeneous or m nonhomogeneous equations in n unknowns ($m \neq n$) is explained in Ref. 17 at the end of the text.

28. Application of Determinants in Obtaining the Particular Integral or Steady-state Solution of Simultaneous Differential Equations with Constant Coefficients and Sinusoidal Applied Force. Let it be required to find the steady-state solution of the system of differential equations

$$\left.\begin{aligned} z_{11}i_1 + \dots + z_{1n}i_n &= E \sin \omega t, \\ z_{21}i_1 + \dots + z_{2n}i_n &= 0, \\ \cdot \qquad \quad \cdot & \\ \cdot \qquad \quad \cdot & \\ \cdot \qquad \quad \cdot & \\ z_{n1}i_1 + \dots + z_{nn}i_n &= 0, \end{aligned}\right\} \tag{126}$$

where

$$z_{rs}i_s = L_{rs}\frac{di_s}{dt} + R_{rs}i_s + \frac{1}{C_{rs}}\int_0^t i_s dt$$

and L_{rs}, R_{rs}, and C_{rs} are constants.

If $\frac{di_s}{dt}$ be denoted by pi_s and $\int_0^t i_s dt$ by $\frac{1}{p}i_s$, then

$$z_{rs}(p) = L_{rs}p + R_{rs} + \frac{1}{C_{rs}p}.$$

The system of Eqs. (126) may represent either a linear electric circuit network with a sinusoidal applied voltage in one mesh or a mechanical system of n degrees of freedom with a sinusoidal applied force somewhere in the system.

The calculation of the particular integral or steady-state solution of such a system is reduced to algebraic computation by the following proof. By subtraction of Eqs. (26) and (27) of § 11, it follows that

$$\sin \omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}.$$

Let us first find the particular integral of the system of equations

$$\left.\begin{aligned} z_{11}i_1 + \ldots + z_{1n}i_n &= \frac{Ee^{i\omega t}}{2i}, \\ z_{21}i_1 + \ldots + z_{2n}i_n &= 0, \\ \cdot \qquad\qquad & \cdot \\ \cdot \qquad\qquad & \cdot \\ \cdot \qquad\qquad & \cdot \\ z_{n1}i_1 + \ldots + z_{nn}i_n &= 0. \end{aligned}\right\} \tag{127}$$

Since the result of $F_{rs}(p)$ operating on a function $e^{i\omega t}$ is a constant times $e^{i\omega t}$ or

$$z_{rs}(p)e^{i\omega t} = \left[L_{rs}i\omega + R_{rs} + \frac{1}{Ci\omega}\right]e^{i\omega t} = \text{constant} \times e^{i\omega t},$$

it is evident that $i_1, \ldots, i_n$ must be expressible as some linear combination of $e^{i\omega t}$ if the first equation of (127) is to be satifised for all values of time. Therefore, let us try as a solution

$$i_s = I_s e^{i\omega t} \quad (s = 1, 2, \ldots, n). \tag{128}$$

For abbreviation let

$$L_{rs}i\omega + R_{rs} + \frac{1}{C_{rs}i\omega} = \lambda_{rs}.$$

Substituting (128) in (127), we have

$$\left.\begin{aligned} \lambda_{11}I_1 + \ldots + \lambda_{1n}I_n &= \frac{E}{2i}, \\ \lambda_{21}I_1 + \ldots + \lambda_{2n}I_n &= 0, \\ & \\ \lambda_{n1}I_1 + \ldots + \lambda_{nn}I_n &= 0. \end{aligned}\right. \tag{129}$$

Solving by determinants for any I (say I_s), we have

$$I_s = \frac{EA_{1s}(i\omega)}{2i\Delta(i\omega)},$$

where $\Delta(i\omega)$ is the determinant of the coefficients of the system (129) and $A_{1s}(i\omega)$ is the cofactor [11] of λ_{1s} in Δ. If $\dfrac{\Delta(i\omega)}{A_{1s}(i\omega)}$ be denoted by

[11] The cofactor is the minor taken with the algebraic sign determined by Laplace's expansion.

$Z_{1s}(i\omega)$,

$$i_s = \frac{Ee^{i\omega t}}{2iZ_{1s}(i\omega)}. \tag{130}$$

To obtain the solution of system (127) with $\frac{Ee^{i\omega t}}{2i}$ replaced by $\frac{Ee^{-i\omega t}}{2i}$, it is only necessary to replace i by $-i$ in Eq. (130). Then the particular integral of (126) for I_s is the difference of (130) and (130) with i replaced by $-i$, or

$$i_s = \frac{E}{2i}\left[\frac{e^{i\omega t}}{Z_{1s}(i\omega)} - \frac{e^{-i\omega t}}{Z_{1s}(-i\omega)}\right]. \tag{131}$$

Since $\Delta(i\omega)$ and $A_{1s}(i\omega)$ are both polynomials in $i\omega$, both are complex numbers. Hence $Z_{1s}(i\omega)$ is a complex number (say $a + bi$). The complex number $a + bi$ can be written $re^{i\phi}$ where r is its modulus and ϕ its argument. But since $Z_{1s}(-i\omega)$ is obtained from $Z_{1s}(i\omega)$ by replacing i by $-i$, it follows that $Z_{1s}(-i\omega)$ and $Z_{1s}(i\omega)$ are conjugate complex numbers. Thus if

$$Z_{1s}(i\omega) = re^{i\phi},$$

then

$$Z_{1s}(-i\omega) = re^{-i\phi}.$$

If these values for $Z_{1s}(i\omega)$ and $Z_{1s}(-i\omega)$ are substituted in Eq. (131)

$$i_s = \frac{E}{2ri}[e^{i(\omega t-\phi)} - e^{-i(\omega t-\phi)}] = \frac{E \sin(\omega t - \phi)}{Z_{1s}(i\omega)} \tag{132}$$

where $|Z_{1s}(i\omega)|$ is the absolute value of the complex number $Z_{1s}(+i\omega)$ and ϕ is its argument.

Example. Compute the steady-state terms or particular integrals of the solution of differential equations (106–107).

The symbols of Eqs. (129–132) for the Eqs. (106–107) then are

$$\Delta(i\omega) = \begin{vmatrix} -M_1\omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -M_2\omega^2 + k_2 \end{vmatrix},$$

$$A_{11}(i\omega) = k_2 - M_2\omega^2, \quad A_{12} = k_2, \quad \omega = \frac{\pi V}{L},$$

$$Z_{11}(i\omega) = \frac{\Delta(i\omega)}{A_{11}(i\omega)}, \quad Z_{12}(i\omega) = \frac{\Delta(i\omega)}{A_{12}(i\omega)}.$$

Thus by Eq. (132) the steady-state terms, s_{1s} and s_{2s}, of s_1 and s_2 are, respectively,

$$s_{1s} = \frac{k_1 y_0 (k_2 - M_2\omega^2)\sin(\omega t - \phi)}{Z},$$

$$s_{2s} = \frac{k_1 y_0 k_2 \sin(\omega t - \phi)}{Z},$$

where $\phi = 0$ and $Z = M_1M_2\omega^4 - (M_1k_2 + M_2k_1 + M_2k_2)\omega^2 + k_1k_2$. The values of s_{1s} and s_{2s}, of course, agree with those obtained in Eqs. (109) and (110).

29. Application of Determinants in Obtaining the Complementary Function in the Solution of Simultaneous Differential Equations with Constant Coefficients. We now obtain the complementary functions for Eqs. (106–107) by what is frequently called the classical method. Since the coefficients of the differential equations are constants, we substitute in (106–107)

$$\left.\begin{aligned} s_1 &= C_1e^{mt}, \\ s_2 &= C'_1e^{mt}. \end{aligned}\right\} \qquad (133)$$

Making the substitution (133) and dividing out the factor e^{mt}, we have

$$\begin{aligned} (M_1m^2 + k_1 + k_2)C_1 - k_2C'_1 &= 0, \\ -k_2C_1 + (M_2m^2 + k_2)C'_1 &= 0. \end{aligned} \qquad (134)$$

These equations are of the form (125), and by § 27 there exists a solution of (134) (other than $C_1 = C'_1 = 0$) only in case

$$\begin{vmatrix} M_1m^2 + k_1 + k_2 & -k_2 \\ -k_2 & M_2m^2 + k_2 \end{vmatrix} = 0. \qquad (135)$$

Eq. (135) is called the **characteristic equation** of the system of equations (106–107). Evidently (135), which is Eq. (108*a*), has four roots. In § 18 these roots have been found to be $\pm\,\omega_1 i$, $\pm\,\omega_2 i$. Since Eqs. (106–107) are linear, the complete complementary functions are

$$\begin{aligned} s_1 &= C_1e^{i\omega_1 t} + C_2e^{-i\omega_1 t} + C_3e^{i\omega_2 t} + C_4e^{-i\omega_2 t}, \\ s_2 &= C'_1e^{i\omega_1 t} + C'_2e^{-i\omega_1 t} + C'_3e^{i\omega_2 t} + C'_4e^{-i\omega_2 t}. \end{aligned} \qquad (136)$$

The eight arbitrary constants of Eqs. (136) are not independent. The relations between C_1 and C'_1, C_2 and C'_2, C_3 and C'_3, etc., are given by either of Eqs. (134) when m has been replaced respectively by $\omega_1 i$, $-\omega_1 i$, and $\omega_2 i$, $-\omega_2 i$.

From the first of (134)

$$C_1 = \frac{k_2 C'_1}{-M_1\omega_1^2 + k_1 + k_2}.$$

From the second of (134)

$$C_1 = \frac{(-M_2\omega_1^2 + k_2)C'_1}{k_2}.$$

The last two relations are identical provided

$$\frac{k_2}{-M_1\omega_1^2 + k_1 + k_2} = \frac{-M_2\omega_1^2 + k_2}{k_2}.$$

But this equation is only the characteristic equation, with $m = \omega_1 i$, written in different form, and hence is true. C_2 is related to C'_2 in the same way that C_1 is to C'_1. The relation between C_3 and C'_3 is either

$$C_3 = \frac{k_2 C'_3}{-M_1\omega_2^2 + k_1 + k_2},$$

or

$$C_3 = \frac{(-M_2\omega_2^2 + k_2)C'_3}{k_2}.$$

The same relationship holds between C_4 and C'_4. The imaginaries are eliminated from (136) by Eqs. (26) and (27).

The method just illustrated is applicable to all systems of the form (126). The characteristic equation in this case is

$$\begin{vmatrix} \lambda_{11}\lambda_{12} \cdots \lambda_{1n} \\ \cdots\cdots \\ \lambda_{n1}\lambda_{n2} \cdots \lambda_{nn} \end{vmatrix} = 0, \text{ where } \lambda_{rs} = L_{rs}m + R_{rs} + \frac{1}{C_{rs}m}.$$

However, if the system of differential equations is at all complicated, the calculation of C'_i, C_i'', C_i''', etc., in terms of C_i along with the evaluation of the arbitrary constants $C_i(i = 1, 2, \ldots n)$ is so laborious that the classical method is practically useless, and recourse is had to the operational calculus of Chap. IV.

Evidently the characteristic equation $\Delta(m) = 0$, in a system of many degrees of freedom, is an algebraic equation of high degree in m. This equation plays an important rôle in the study of electrical and mechanical system. Tests for stability of systems, whose differential equations of motion are linear, are developed in terms of the characteristic equation. (See § 47, problem 6.) The roots of the characteristic equation give at once the natural frequencies of vibration of the system being studied. For this reason, in applications of the operational

calculus, it is frequently necessary to obtain the roots of the characteristic equation. Sec. IV of this chapter is concerned with finding the roots of higher-degree algebraic equations.

EXERCISES AND PROBLEMS

1. Some of the following systems of equations have one or more solutions. Obtain, by use of determinants, these solutions,

(a)
$$\begin{aligned} x + y + z &= 11, \\ 2x - 6y - z &= 0, \\ 3x + 4y + 2z &= 0. \end{aligned}$$

(b)
$$\begin{aligned} x + 2y + 3z + 4w &= 34, \\ -x + 3y + 7z + 2w &= 36, \\ 4x + 8y + 5z + 6w &= 65, \\ -4x + 7y + 3z + 4w &= 39. \end{aligned}$$

(c)
$$\begin{aligned} x + 2y + 3z + 2w &= 34, \\ -x + 3y + 7z - 2w &= 36, \\ 4x + 8y + 5z + 8w &= 65, \\ -4x + 7y + 3z - 8w &= 39. \end{aligned}$$

(d)
$$\begin{aligned} -x - 2y + 3z + 4w &= 0, \\ x - 3y + 7z + 6w &= 0, \\ 4x - 8y + 5z + 16w &= 0, \\ 9x + y + z - 2w &= 0. \end{aligned}$$

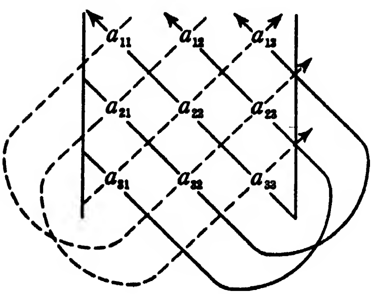
(e)
$$\begin{aligned} x + 2y + 3z + 4w &= 34, \\ 3x + 6y + 9z + 12w &= 102, \\ 4x + 8y + 5z + 6w &= 65, \\ -4x + 7y + 3z + 4w &= 39. \end{aligned}$$

(f)
$$\begin{aligned} x + 2y + 3z + 4w &= 0, \\ -x + 3y + 7z + 2w &= 0, \\ 4x + 8y + 5z + 6w &= 0, \\ -4x + 7y + 3z + 4w &= 0. \end{aligned}$$

2. Write the characteristic determinant for each of the systems of Eqs. (114) (115), and (116).

3. Compute the steady-state solution for each of the systems of Eqs. (114), (115), and (116).

4. In the case of a third-order determinant, Laplace's expansion is equivalent to adding the products indicated by the heavy lines and subtracting the products indicated by the dotted lines. That is



$$= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{23}a_{12} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{32}a_{23}.$$

Show that this graphical method fails for determinants of order higher than 3 unless certain relations hold between the elements. In a determinant of the fourth order, write down the relations that must hold between the elements in order that this method may give the same result as Laplace's expansion.

III

FOURIER SERIES

30. Definitions. A series of the form

$$\frac{a_0}{2} + \sum^{\infty} (a_n \cos nt + b_n \sin nt), \tag{137}$$

where a_n, b_n are independent of t, is called a **trigonometric series.** The symbol t may represent time or any other variable. The constant term is written as $\frac{a_0}{2}$ instead of a_0 for a reason that will be explained in § 33. If a_n, b_n are given by certain formulas (derived in § 33), then the series (137) is a **Fourier series** and a_n, b_n are called Fourier coefficients. To obtain a Fourier series which shall represent a function $y = f(t)$ for every value of t within the interval $t_1 < t < t_2$, it is only necessary to compute the Fourier coefficients and substitute these values in series (137).

Most (but not all) single-valued functions of applied mathematics can be expanded in a Fourier series. Any function $y = f(t)$ defined in the interval $t_1 < t < t_2$ can be expanded in a Fourier series provided that in this interval:

(a) y is not infinite,
(b) there is exactly one value of y for every value of t,
(c) y has only a finite number of maxima or minima,
(d) y has not more than a finite number of finite discontinuities.

Functions having the properties just enumerated occur so frequently in engineering that it is an economy of words and time to label such

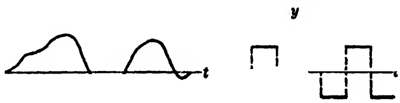


FIG. 17.

functions **engineering functions.** In the remainder of the text, they will be so denoted.

In order to find the Fourier series representing an engineering function $y = f(t)$ it is not necessary that the function be expressed by means of an equation. The functional relation may be given by means of a continuous curve or by a series of broken curves whose equations may or may not be known. The functions represented in Fig. 17 are typical engineering functions for expansion in Fourier series.

31. Introductory Problem to Illustrate the Use of the Fourier Series. In § 16 reference was made to obtaining the particular integral of the differential equation $F(p)y = f(t)$ by use of Fourier series. Suppose that the mass in the system described in § 10*a* and illustrated in Fig. 2 is acted upon by an external periodic force $f(t)$, positive downward, whose graph is shown in Fig. 18. The force is applie to the mass beginning at time $t = 0$. Let it be required to determine the motion of the mass. As in § 10, D'Alembert's principle is em-

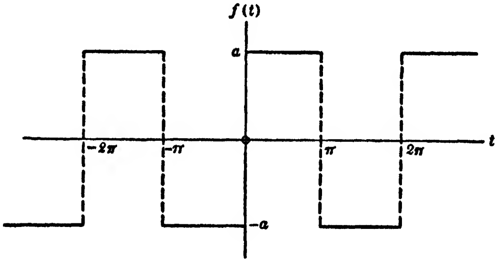


FIG. 18.

ployed, and the force $f(t)$ is included as well as the others. The resulting differential equation becomes

$$M\frac{d^2s}{dt^2} + k_d\frac{ds}{dt} + ks = f(t).$$

It will be shown later that the Fourier development of the function in Fig. 18 is

$$f(t) = \frac{4a}{\pi}[\sin t + \tfrac{1}{3}\sin 3t + \tfrac{1}{5}\sin 5t + \dots],$$

$$= \frac{4a}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)t.$$

Consequently the differential equation of motion of the mass is

$$M\frac{d^2s}{dt^2} + k_d\frac{ds}{dt} + ks = \frac{4a}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)t. \qquad (138)$$

By use of the method of § 28, the value of u as a function of the time is

$$u = \frac{4a}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\frac{\sin[(2n-1)t - \phi_n]}{\{[k - M(2n-1)^2]^2 + k_d^2(2n-1)^2\}^{1/2}}, \qquad (139)$$

where

$$\phi_n = \tan^{-1} \frac{k_d(2n - 1)}{k - M(2n - 1)^2}.$$

The complete solution of (138) is

$$s = C_1 e^{-k_d t/2M} \sin (\omega t - \phi) + u,$$

where $\omega = \dfrac{\sqrt{4kM - k_d^2}}{2M}$, C_1 and ϕ are arbitrary constants which depend upon the initial conditions and u is given by Eq. (139). In this case, the force function was represented by a Fourier series to facilitate the solution.

32. A Second Introductory Problem. Very similar to (138) is the differential equation describing the motion of the rotor of a synchronous generator which is driven by a reciprocating engine. Let us determine this differential equation, using the principle employed in § 10*b*, namely, that the algebraic sum of the torques acting on the rotor is zero.

The torques acting are:

(*a*) torque due to inertial reaction $= - I \dfrac{d^2\theta}{dt^2}$,

(*b*) synchronizing torque $= T_s(\omega t - \theta)$,

(*c*) damping torque $= T_d \dfrac{d}{dt} (\omega t - \theta)$,

(*d*) applied torque of the reciprocating engine $= f(t)$.

The symbols employed are defined as follows: I is the moment of inertia of the rotor and shaft; T_s is the synchronizing torque coefficient; T_d the damping torque coefficient due to amortisseur windings, eddy currents, etc.; ω is the synchronous speed in mechanical radians per second; and θ is the angular displacement of the rotor at time t, measured from a stationary reference. From a mathematical point of view, it is only necessary to know that I, T, T_d and ω are constants. For more complete engineering information, see Ref. 19 at the end of the text. The differential equation of motion of the rotor is thus

$$-I \frac{d^2\theta}{dt^2} + T_d \frac{d}{dt} (\omega t - \theta) + T_s(\omega t - \theta) + f(t) = 0. \qquad (140)$$

The engine torque $f(t)$ is, of course, variable. It is represented by some curve like that of Fig. 19, in which the line y_0 represents the average torque.

If $f(t)$ is developed in a Fourier series by the method of § 33, then Eq. (140) can be solved by the method used in the solution of (138).

The applications of Fourier series in the two above introductory problems alone would justify a study of such series from an engineering point of view. In these two applications, the applied forces are periodic

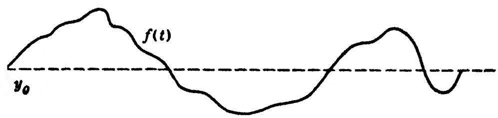


FIG. 19.

functions of the time. Also, the series of Eq. (137) is periodic, of period 2π since

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos n(t + 2\pi) + b_n \sin n(t + 2\pi)]$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nt + b_n \sin nt].$$

Consequently, it is reasonable to suppose that Fourier series are especially suitable for the expansion of periodic functions, and such is the case. The application of Fourier series is not limited to the expansion of periodic functions of period 2π, but as shown in § 36 they are also applicable to periodic functions of period $2l$ where l is any positive constant. Since voltage, current, flux density, permeance coefficient, applied force, motion, potential, and magnetomotive force are often periodic, the applications of Fourier series are very numerous. Nonperiodic functions are also expansible in a given interval in Fourier series. Such series also find extensive use in the solution of partial differential equations, as will be explained in a later chapter.

33. Values of the Fourier Coefficients a_n, b_n. We now obtain formulas for the calculation of the coefficients a_n, b_n of the Fourier series representing a given function when the functional relation is given by an equation. In § 37 we shall derive the Fourier coefficients in case the functional relation is given by a graph. Let it be assumed that the engineering function $y = f(t)$, given in the interval $(-\pi, \pi)$, may be written as the series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nt + b_n \sin nt]. \tag{141}$$

To evaluate the a_n's, which are independent of t, in (141) multiply both sides of this equation by $\cos mt$ and integrate from $-\pi$ to π. Since

$$\int_{-\pi}^{\pi} \cos mt \cos nt dt = \int_{-\pi}^{\pi} \sin mt \sin nt dt \begin{cases} = 0, & m \neq n \\ = \pi, & m = n \neq 0 \end{cases}$$

and

$$\int_{-\pi}^{\pi} \sin mt \cos nt dt = 0,$$

we obtain

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt dt, \tag{142}$$

and for the special case of $n = 0$, $\cos nt = 1$ and

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) dt. \tag{142a}$$

Similarly, using $\sin mt$ as a multiplier,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt dt. \tag{143}$$

Now substitute these expressions for the coefficients in the assumed series, Eq. (141). However, to avoid confusing the variable of integration t in the definite integrals (142), (142a), and (143) with the variable t in (141), let us first replace t by t' in (142), (142a), and (143). It is legitimate to change the variable of integration in a definite integral, for in the substitution of limits this variable disappears and hence the definite integral is not a function of the variable of integration. Then (141) becomes

$$f(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t')dt' + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\left[\int_{-\pi}^{\pi} f(t') \cos nt'dt'\right] \cos nt\right)$$
$$+ \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\left[\int_{-\pi}^{\pi} f(t') \sin nt'dt'\right] \sin nt\right). \tag{144}$$

This is the Fourier series for $f(t)$ in the interval $-\pi$ to π.

It may be seen that by writing the constant term of (137) as $\frac{a_0}{2}$ instead of, say, a_0 the equation determining it, (142a), is a special case of (142) with $n = 0$. Otherwise (142a) would differ from the special case of (142) by a factor of $\frac{1}{2}$.

EXAMPLE 1. Expand $f(t) = t^2$ in a Fourier series where t lies in the interval $-\pi$ to π. Substituting in formulas (142) and (143), we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos nt dt = \frac{4}{n^2}\cos n\pi (n \neq 0),$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \sin nt dt = 0, \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 dt = \frac{2\pi^2}{3}.$$

If these values of a_n and b_n are substituted in Eq. (141) the Fourier expansion for t^2 is

$$t^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos t}{1} - \frac{\cos 2t}{2^2} + \frac{\cos 3t}{3^2} - \frac{\cos 4t}{4^2} + \cdots\right]$$

The expansion just obtained represents t^2 within the interval $-\pi<t<\pi$. Outside this interval, t^2 is not represented by the above series. This property is true in general when a non-periodic function is expanded in a Fourier series. When a periodic continuous function is expanded in a Fourier series, the function is represented by the Fourier series for all values of the independent variable.

EXAMPLE 2. Develop in a Fourier series the function

$$\begin{aligned} f(t) &= a \text{ for } -2\pi < t < -\pi \\ f(t) &= -a \text{ for } -\pi < t < 0 \\ f(t) &= a \text{ for } 0 < t < \pi \\ f(t) &= -a \text{ for } \pi < t < 2\pi \end{aligned}$$

whose graph is shown in Fig. 18.

Formulas (142) and (143) give

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t') \cos nt'dt' = \frac{1}{\pi}\int_{-\pi}^{0} (-a) \cos nt'dt' + \frac{1}{\pi}\int_{0}^{\pi} a \cos nt'dt' = 0,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t') \sin nt'dt' = \frac{1}{\pi}\int_{-\pi}^{0} (-a) \sin nt'dt' \quad \frac{1}{\pi}\int_{0}^{\pi} a \sin nt'dt' = \frac{2a}{n\pi}(1 - \cos n\pi).$$

When these values of a_n and b_n are substituted in (141), the Fourier series for the function of this example is

$$f(t) = \frac{4a}{\pi}[\sin t + \tfrac{1}{3}\sin 3t + \tfrac{1}{5}\sin 5t \ldots]$$

$$= \frac{4a}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)t.$$

This series represents the function of Fig. 18 for all values of t from $-n\pi$ to $+n\pi$ except those values which render the function discontinuous. At points of discontinuity, the series gives the value zero, which is the mean of the two values of the function as the point of discontinuity is approached from the right and from the left. This illustrates the general result that at a point (say, $t = t_1$) of finite discontinuity of $f(t)$ the Fourier series gives the value

$$\tfrac{1}{2}\left[\lim_{\varepsilon\to 0} f(t_1 - \varepsilon) + \lim_{\varepsilon\to 0} f(t_1 + \varepsilon)\right]$$

In example 1 the Fourier series contains only cosine terms (and a constant). This is due to the fact that $y = t^2$ is an **even function.** A function $y = f(t)$ is even if $f(-t) = f(t)$. The series in example 2 contains only sine terms since the function is an **odd function.** That is, $f(-t) = -f(t)$. The Fourier expansion in the interval $-\pi$ to π of a function which is neither even nor odd (for example $y = e^t$) contains both sine and cosine terms and possibly also a constant.

34. Fourier Series Expansion for the Interval 0 to 2π. If the expansion of $y = f(t)$ is desired for the interval 0 to 2π, it can be easily shown by the method of § 33 that the coefficients a_n, b_n are

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t')dt', \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(t')\cos nt'dt',$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t')\sin nt'dt'. \tag{145}$$

35. Fourier Expansions in Sines or Cosines Only. In the interval 0 to π, it is possible to expand an engineering function $f(t)$ either in a sine series or in a cosine series or in a sine and cosine series. The coefficients for a cosine series, in which the constant term is $\frac{a_0}{2}$, are:

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(t')\cos nt'dt', \quad b_n = 0. \tag{146}$$

The coefficients for a sine series are:

$$a_n = 0,\ b_n = \frac{2}{\pi}\int_0^{\pi} f(t') \sin nt'dt'. \tag{147}$$

The proof that the coefficients for the cosine series are given by (146) is as follows. Define a new engineering function $y = F(t)$ such that $F(t) = f(t)$ for $0 < t < \pi$ and $F(t) = f(-t)$ for $-\pi < t < 0$. Thus $y = F(t)$ is an even function. Expanding $F(t)$ in a Fourier series by formulas (142) and (143), we have:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t') \cos nt'dt' = \frac{1}{\pi}\int_{-\pi}^{0} f(-t') \cos nt'dt' + \frac{1}{\pi}\int_0^{\pi} f(t') \cos nt'dt'.$$

But since $F(-t) = F(t)$

$$\int_{-\pi}^{0} f(-t') \cos nt'dt' = \int_0^{\pi} f(t') \cos nt'dt'.$$

Consequently,

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(t') \cos nt'dt'.$$

Likewise

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t') \sin nt'dt' = \frac{1}{\pi}\int_{-\pi}^{0} f(-t') \sin nt'dt + \frac{1}{\pi}\int_0^{\pi} f(t') \sin nt'dt.$$

But since $F(-t') = F(t)$

$$\int_{-\pi}^{0} f(-t') \sin nt'dt' = -\int_0^{\pi} f(t') \sin nt'dt$$

and

$$b_n = 0.$$

Since $F(t)$ was defined as equal to $f(t)$ in the interval 0 to π, the series for $F(t)$ also represents $f(t)$ in that interval.

Example. Expand $y = \sin t$ in a cosine series valid for the interval 0 to π. The application of formulas (146) gives

$$\sin t = \frac{4}{\pi}\left[\frac{1}{2} - \frac{\cos 2t}{1.3} - \frac{\cos 4t}{3.5} - \frac{\cos 6t}{5.7} - \cdots\right].$$

The derivation of formulas (147) is similar to the derivation of (146).

The expansion of $y = f(t)$, defined in the interval *0 to* π, in a sine and cosine series is not unique. (That is, there is more than one expansion in both sines and cosines which represents the function.) To see that this is true, define a new function $y = F(t)$ such that $F(t) = \phi(t)$ for $-\pi < t < 0$ and $F(t) = f(t)$ for $0 < t < \pi$. Let $\phi(t)$ be such that $F(t)$ is neither an even nor an odd function. By § 33 the expansion of $F(t)$ for the interval $-\pi$ to π will contain both sine and cosine terms. It will represent $F(t)$ in the interval $-\pi$ to π and hence $f(t)$ in the interval 0 to π. Since there is an infinitude of functions satisfying the restrictions placed on $\phi(t)$ and since the coefficients a_n and b_n depend upon $\phi(t)$ as well as $f(t)$, the expansion is not unique.

The Fourier expansion of an engineering function for the interval $-\pi$ to π is unique.

36. Fourier Series for the Interval $-l$ to l. The expansion of a function in a Fourier series may be extended from the interval $(-\pi, \pi)$ to the interval $(-l, l)$, where l is any positive constant. Suppose that it is desired to expand an engineering function $f(y)$ in a Fourier series between the limits $y = -l$ to $y = l$. Consider another variable t such that $t = -\pi$ when $y = -l$ and $t = \pi$ when $y = l$. That is, t may be related to y by the equation

$$t = \frac{\pi y}{l} \quad \text{or} \quad y = \frac{lt}{\pi},$$

from which

$$dt = \frac{\pi}{l}\,dy.$$

Now considering $f(y)$ to be a function of t, that is, $f\left(\frac{lt}{\pi}\right)$, it may be expanded by Eq. (144) into the form

$$f(y) = f\left(\frac{lt}{\pi}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{lt'}{\pi}\right) dt'$$

$$+ \frac{1}{\pi}\sum_{1}^{\infty}\left(\left[\int_{-\pi}^{\pi} f\left(\frac{lt'}{\pi}\right)\cos nt'dt'\right]\cos nt\right)$$

$$+ \frac{1}{\pi}\sum_{1}^{\infty}\left(\left[\int_{-\pi}^{\pi} f\left(\frac{lt'}{\pi}\right)\sin nt'dt'\right]\sin nt\right).$$

Making use of the substitution $\frac{lt}{\pi} = y$, we have

$$
\begin{aligned}
f(y) = \frac{1}{2l}\int_{-l}^{l} f(y')dy' \\
+\frac{1}{l}\sum_{1}^{\infty}\left(\left[\int_{-l}^{l} f(y')\cos\left(\frac{n\pi y'}{l}\right)dy'\right]\cos\left(\frac{n\pi y}{l}\right)\right) \\
+\frac{1}{l}\sum_{1}^{\infty}\left(\left[\int_{-l}^{l} f(y')\sin\left(\frac{n\pi y'}{l}\right)dy'\right]\sin\left(\frac{n\pi y}{l}\right)\right). \qquad (148)
\end{aligned}
$$

This is the expansion desired.

Similarly, by means of § 35 and the first part of this article, it can be shown that for the interval 0 to l the expansions of sines and cosines only are, respectively:

$$f(t) = \frac{2}{l}\sum_{1}^{\infty}\left(\left[\int_{0}^{l} f(t')\sin\frac{n\pi t'}{l}dt'\right]\sin\frac{n\pi t}{l}\right), \qquad (149)$$

$$f(t) = \frac{1}{l}\int_{0}^{l} f(t')dt' + \frac{2}{l}\sum_{1}^{\infty}\left(\left[\int_{0}^{l} f(t')\cos\frac{n\pi t'}{l}dt\right]\cos\frac{n\pi t}{l}\right). \qquad (150)$$

37. Harmonic Analysis. Suppose that the function to be represented is given by a graph, as in the case of an oscillogram, or by a table of values relating dependent and independent variables. The determination of the constants a'_n, b'_n in the series

$$a'_0 + \sum^{\infty} a'_n \cos nt + b'_n \sin nt,$$

so that the series may represent the function, is known as **harmonic analysis.** The primes distinguish the coefficients determined graphically from those determined by integration. There are three cases of most frequent occurrence in engineering for which this determination is desired. The procedure in the three cases is as follows:

Case 1. **If the curve or table of values repeats itself at equal intervals of the independent variable, consider only one of the intervals or sections.**

(a) Choose the unit so that the length of the section is 2π.

(b) Divide the section into k equal intervals each of width Δt by introducing the $(k - 1)$ interior points $t_1, t_2, \ldots, t_{k-1}$.

(c) To determine the coefficient b'_n $(n = 1, 2, 3, \ldots)$ of any sine term, take the average of the k ordinates after having multiplied each by the sine of n times the angle at which it is taken. The k ordinates are either those whose abscissas are t_1, t_2, t_3, . . . and 2π or 0, t_1, t_2, . . . t_{k-1}. Care must be taken not to take the ordinates at both 0 and 2π. The reason for this becomes evident in the proof of these rules in § 38. The coefficient b'_n is twice this average.

(d) To determine the coefficient a'_n $(n = 0, 1, 2, 3 \ldots)$ of any cosine term take the average of the k ordinates after having multiplied each by the cosine of n times the angle at which it is taken. The value of a'_n is twice this average.

Case 2. **If the curve or table of values does not repeat itself, consider only the section of the curve over which it is desired to make calculations. The steps to be followed are precisely (a), (b), (c), and (d) of case (1).**

Case 3. **The Ordinates of oscillograms of steady-state alternating currents in most electrical machinery not only repeat themselves at intervals of 2π, but also reverse in sign at intervals of π. In other words, if the functional relation of such a curve is $i = f(t)$ then $f(t + \pi) = - f(t)$. The curve in Fig. 20 is an example. In the Fourier representation of such a curve only those harmonics (sines and cosines) can occur which reverse in sign when t is increased by π. Consequently, only odd harmonics are present. This observation reduces the number of calculations one-half. The procedure in this case is identical to that of case 1 except that the calculation is made for only one arch of the curve. The unit is so chosen that the length of this arch is π. The coefficients of all even harmonics are zero.**

Before proving the rules just given, we illustrate their application.

EXAMPLE 1. Let the function $y = f(t)$ be given by the table of values:

$$\begin{array}{lccccccccc}
t = & 0, & \frac{\pi}{4}, & \frac{\pi}{2}, & \frac{3\pi}{4}, & \pi, & \frac{5\pi}{4}, & \frac{3\pi}{2}, & \frac{7\pi}{4}, & 2\pi \\
f(t) & 2, & \frac{3}{2}, & 1, & \frac{1}{2}, & 0, & \frac{1}{2}, & 1, & \frac{3}{2}, & 2
\end{array}$$

$$\begin{array}{lcccccccccc}
 & \frac{9\pi}{4}, & \frac{5\pi}{2}, & \frac{11\pi}{4}, & 3\pi, & \frac{13\pi}{4}, & \frac{7\pi}{2}, & \frac{15\pi}{4}, & 4\pi, & \frac{17\pi}{4}, & \text{etc.} \\
f(t) = & \frac{3}{2}, & 1, & \frac{1}{2}, & 0, & \frac{1}{2}, & 1, & \frac{3}{2}, & 2, & \frac{3}{2}, & \text{etc.}
\end{array}$$

Let us make an harmonic analysis of this function.

Evidently from the table $f(2\pi - t) = f(t)$. Thus no sine terms can appear since $\sin n(2\pi - t) \neq \sin nt$. Since $\cos n(2\pi - t) = \cos nt$

for $n = 0$ or a positive integer, all cosine terms may be present. The function is periodic, hence only one period of 2π need be used. The calculations are best carried out in tabular form. (No confusion will result if the primes are omitted from a'_n and b'_n in computations.)

t	y	cos t	y cos t	cos $2t$	y cos $2t$	cos $3t$	y cos $3t$
$\frac{\pi}{4}$	$\frac{3}{2}$	0.707	1.060	0	0	−0.707	−1.060
$\frac{\pi}{2}$	1	0.000	0.000	−1	−1	0.000	0.000
$\frac{3\pi}{4}$	$\frac{1}{2}$	−0.707	−0.354	0	0	0.707	0.354
π	0	−1.000	0.000	1	0	−1.000	0.000
$\frac{5\pi}{4}$	$\frac{1}{2}$	−0.707	−0.354	0	0	0.707	0.354
$\frac{3\pi}{2}$	1	0.000	0.000	−1	−1	0.000	0.000
$\frac{7\pi}{4}$	$\frac{3}{2}$	0.707	1.060	0	0	−0.707	−1.060
2π	2	1.000	2.000	1	2	1.000	2.000
......	8	....	3.412	....	0	....	0.588

$$a_0 = 2 \times \text{average of the } y \text{ column} = 2.$$

$$a_1 = 2 \times \text{average of } y \cos t \text{ column} = 0.853.$$

$$a_2 = 2 \times \text{average of } y \cos 2t \text{ column} = 0.$$

$$a_3 = 2 \times \text{average of } y \cos 3t \text{ column} = 0.147.$$

Thus $y = f(t) = 1 + 0.853 \cos t + 0.147 \cos 3t + \dots$. The coordinates of every point (t, y) of the given function $y = f(t)$ satisfy this equation approximately.

Example 2. In a later chapter (Vol. II, Chap. II) on the solution of non-linear differential equations, an harmonic analysis of an oscillogram of the current will need to be made. The series so obtained will be a check on the answer there obtained by the analytical solution of the non-linear circuit equation. Fig. 20 is the reproduction of such an oscillogram. By use of the method of § 37 an harmonic analysis will be made for this graph. Evidently, by case 3, § 37, no even harmonics are present. The computations in tabular form are as follows. The numerical values of i in the table are given in hundredths of an ampere.

θ	i	$\sin\theta$	$\cos\theta$	$i\sin\theta$	$i\cos\theta$	$\sin 3\theta$	$\cos 3\theta$	$i\sin 3\theta$	$i\cos 3\theta$	$\sin 5\theta$	$\cos 5\theta$	$i\sin 5\theta$	$i\cos 5\theta$
0°	0.00	0.000	1.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	1.000	0.000	0.000
10°	0.52	0.174	0.985	0.090	0.512	0.500	0.866	0.260	0.450	0.766	0.643	0.398	0.334
20°	0.78	0.342	0.940	0.267	0.733	0.866	0.500	0.675	0.390	0.985	−0.174	0.768	−0.136
30°	1.04	0.500	0.866	0.520	0.901	1.000	0.000	1.040	0.000	0.500	−0.866	0.520	−0.901
40°	1.17	0.643	0.766	0.752	0.896	0.866	−0.500	1.013	−0.585	−0.342	−0.940	−0.400	−1.100
50°	1.30	0.766	0.643	0.996	0.836	0.500	−0.866	0.650	−1.126	−0.940	−0.342	−1.222	−0.445
60°	1.69	0.866	0.500	1.464	0.845	0.000	−1.000	0.000	−1.690	−0.866	0.500	−1.464	0.845
70°	1.95	0.940	0.342	1.833	0.667	−0.500	−0.866	−0.975	−1.689	−0.174	0.985	−0.339	1.921
80°	2.34	0.985	0.174	2.305	0.407	−0.866	−0.500	−2.026	−1.170	0.643	0.766	1.505	1.792
90°	2.99	1.000	0.000	2.990	0.000	−1.000	0.000	−2.990	0.000	1.000	0.000	2.990	0.000
100°	3.64	0.985	−0.174	3.585	−0.633	−0.866	0.500	−3.152	1.820	0.643	−0.766	2.341	−2.788
110°	4.16	0.940	−0.342	3.910	−1.423	−0.500	0.866	−2.080	3.603	−0.174	−0.985	−0.724	−4.098
120°	4.55	0.866	−0.500	3.940	−2.275	0.000	1.000	0.000	4.550	−0.866	−0.500	−3.940	−2.275
130°	4.16	0.766	−0.643	3.187	−2.675	0.500	0 866	2.080	3.603	−0.940	0.342	−3.910	1.423
140°	2.99	0.643	−0.766	1.923	−2.290	0.866	0.500	2.589	1.495	−0.342	0.940	−1.023	2.811
150°	1.95	0.500	−0.866	0.975	−1.689	1.000	0.000	1.950	0.000	0.500	0.866	0.975	1.689
160°	1.04	0.342	−0.940	0.356	−0.978	0.866	−0.500	0.901	−0.520	0.985	0.174	1.024	0.181
170°	0.39	0.174	−0.985	0.069	−0.384	0.500	−0.866	0.195	−0.338	0.766	−0.643	0.299	−0.251
.....	......	....	....	29.162	−6 550	....	....	0 130	8.793	...	.....	−2.202	−0.999

$$b_1 = 2 \times \text{average of column } i\sin\theta = 2\,\frac{(29.162)}{18} = 3.240.$$

$$a_1 = 2 \times \text{average of column } i\cos\theta = 2\,\frac{(-6.550)}{18} = -\,0.728.$$

$$b_3 = 2 \times \text{average of column } i\sin 3\theta = 2\,\frac{(0.130)}{18} = 0.014.$$

$$a_3 = 2 \times \text{average of column } i\cos 3\theta = 2\,\frac{(8.793)}{18} = 0.977.$$

$$b_5 = 2 \times \text{average of column } i\sin 5\theta$$

$$= 2\,\frac{(-2.202)}{18} = -\,0.245.$$

$$a_5 = 2 \times \text{average of column } i\cos 5\theta$$

$$= 2\,\frac{(-0.999)}{18} = -\,0.111.$$

Thus the Fourier expression for i is

$$i = 3.240 \sin \theta - 0.728 \cos \theta + 0.014 \sin 3\theta + 0.977 \cos 3\theta$$
$$- 0.245 \sin 5\theta - 0.111 \cos 5\theta + \dots$$

or, what is the same thing,

$$i = 3.321 \sin (\theta - 12°40') + 0.977 \sin 3(\theta + 26°57')$$
$$+ 0.269 \sin 5(\theta - 31°28') + \dots$$

Approximate values computed by means of this series are marked by the center of the small circles in Fig. 20. The higher harmonics are seen to be negligible from an engineering point of view.

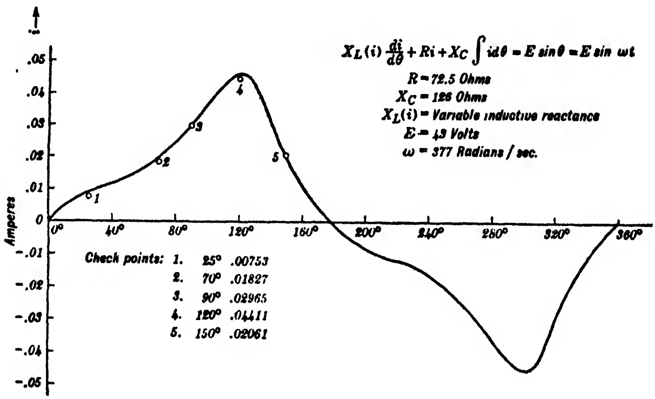


FIG. 20.—Current Non-linear Circuit.

38. Proof of Harmonic Analysis Rules. The procedure outlined in § 37 for finding the Fourier coefficients from a set of points or from a curve is that of integrating graphically the expressions for the coefficients. By Eq. (145)

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t') \sin nt'dt'.$$

But a definite integral may be defined as

$$\int_a^b f(t)dt = \lim_{\substack{k\to\infty \\ \Delta t\to 0}} \sum_{i=1}^{i=k} f(t_i)(\Delta t)_i,$$

where the range $b - a$ of the variable t is divided into k segments

$b - a$

$(\Delta t)_i$ and t_i is any value of t within or at the end points of the ith segment. From the above, we may write

$$
\begin{aligned}
b'_n &= \lim_{\Delta t \to 0} \frac{1}{\pi} \sum_{i=1}^{k} f(i\Delta t) \sin (in\Delta t) \Delta t, \\
&= \text{(approximately, for } \Delta_t \text{ small)} \frac{2}{\frac{2\pi}{\Delta t}} \sum_{i=1}^{k} f(i\Delta t) \sin (in\Delta t) \\
&= \frac{2}{k} \sum_{i=1}^{k} f(i\Delta t) \sin (in\Delta t) \quad (n = 1, 2, 3 \ldots).
\end{aligned}
$$

The last formula written is the expression for b'_n. We have thus provided proof for the first three rules of case 1, § 37. Similarly.

$$
a'_n = \frac{2}{k} \sum_{i=1} f(i\Delta t) \cos (in\Delta t) \quad (n = 1, 2, 3 \ldots),
$$

$$
a'_0 = \frac{2}{k} \sum_{i=1}^{i=k} f(i\Delta t).
$$

The proof of the rule in case 3 is Ex. 1 at the end of Sec. III on Fourier series.

From the definition of a definite integral, it is evident that the ordinates used in the numerical integrations may be taken at any point of the corresponding interval. However, it is simplest to take either the first points or the last points of all the intervals, that is, either $0, t_1, t_2, \ldots t_{k-1}$ or $t_1, t_2, \ldots t_{k-1}, 2\pi$.

There are various ways of making harmonic analyses. The method of §§ 37–38 is as simple as any and at the same time displays clearly the theory of the process. Other methods are described in Ref. 22. If many analyses are to be made, machines called harmonic analyzers are employed.

39. Theory of Fourier Series. Although attention has been focused primarily upon the expansion of engineering functions (defined in § 30) in Fourier series, it is by no means implied that these functions are the most general ones which can be expanded in Fourier series. Much more general functions can be so represented. For example, the function $f(t)$ to be expanded can, under proper restrictions, have an infinitude of discontinuities within a finite interval of t. The function may also have a finite number of points at which it becomes

infinite provided it becomes infinite in certain definite ways. The most general function which can be represented by a Fourier series is unknown. Consequently, at most, it is possible to state only sufficient conditions regarding the expansion of functions in Fourier series. Sufficient conditions, if they are not also necessary conditions, restrict the functions considered to a smaller class.

The treatment of Fourier series thus far has been concerned with the formal expansion of engineering functions. It is the purpose of this section to state and illustrate some of the facts and theorems regarding (*a*) convergence, (*b*) differentiation and integration, and (*c*) manipulation of Fourier series.

(*a*) *Convergence.* The definition of convergence of an infinite series of real continuous functions is readily recalled from the calculus. Let

$$u_1(t) + u_2(t) + u_3(t) + \dots + u_n(t) + \dots$$

be an infinite series of real continuous functions. Let

$$S_n(t) = u_1(t) + u_2(t) + \dots + u_n(t)$$

be the sum of the first n terms of the infinite series. If for some value of t(say t_1) the sum $S_n(t_1)$ approaches a limit $S(t_1)$ as n approaches infinity, then the infinite series in question is said to be **convergent** at $t = t_1$. If a series is not convergent, it is said to be **divergent**. If the series converges for every value of t in the interval $a \leqq t \leqq b$, the series is said to be convergent in the closed interval (a, b). The interval $a < t < b$ is an open interval.

The four following theorems regarding the convergence of Fourier series are important.

1. The Fourier series representing the engineering function $f(t)$ in the interval $-\pi < t < \pi$ converges to the value $f(t)$ at any point t in the interval $-\pi < t < \pi$ at which $f(t)$ is continuous; it converges to the value

$$\tfrac{1}{2}[f(t + 0) + f(t - 0)]$$

at any point at which $f(t)$ is discontinuous. At the end points of the interval $(-\pi, \pi)$ the series converges to the value

$$\tfrac{1}{2}[f(-\pi + 0) + f(\pi - 0)].$$

The symbols

$$\tfrac{1}{2}[f(t + 0) + f(t - 0)]$$

denote

$$\lim_{\varepsilon \to 0} \tfrac{1}{2}[f(t + \varepsilon) + f(t - \varepsilon)].$$

To illustrate this theorem, we refer to § 33, example 2. Let us extend the definition of the function of this example by defining

$$f(-2\pi) = f(-\pi) = f(0) = f(\pi) = f(2\pi) = 0.$$

By the above theorem the series

$$\frac{4a}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m-1}\sin(2m-1)t$$

converges to the value of the function at every point t of the interval $-2\pi \leqq t \leqq 2\pi$.

2. If $f(t)$ is an engineering function in the interval $-\pi < t < \pi$, the coefficients in the Fourier expansion of $f(t)$ are less in absolute value (numerical value, regardless of sign) than C/n, where C is some positive constant independent of n. To illustrate, in the expansion of the function in example 2 of § 33 the coefficients

$$\frac{4a}{\pi}, 0, \frac{4a}{3\pi}, 0, \frac{4a}{5\pi}, \cdots \frac{4a}{(2m-1)\pi},$$

are respectively less than

$$\frac{C}{1}, \frac{C}{2}, \frac{C}{3}, \frac{C}{4}, \frac{C}{5}, \cdots, \frac{C}{2m-1},$$

where C is any number greater than $4a/\pi$ such as $8a/\pi$.

3. If $f(t)$ is a *continuous* engineering function in the interval $-\pi < t < \pi$ and $f(\pi - 0) = f(-\pi + 0)$ and if the derivative $f'(t)$ is an engineering function in the same interval, then the coefficients in the Fourier expansion of $f(t)$ are less in absolute value than C/n^2, where C is some positive constant independent of n. For example, consider the function

$$\begin{aligned} f(t) &= -\pi - t \quad &&\text{for} \quad -\pi < t \leqq -\frac{\pi}{2} \\ f(t) &= t &&\text{for} \quad -\frac{\pi}{2} \leqq t \leqq \frac{\pi}{2} \\ f(t) &= \pi - t &&\text{for} \quad \frac{\pi}{2} \leqq t < \pi \end{aligned}$$

which satisfies the above conditions. Its Fourier expansion is

$$f(t) = \frac{4}{\pi}\left[\sin t - \frac{1}{3^2}\sin 3t + \frac{1}{5^2}\sin 5t - \frac{1}{7^2}\sin 7t + \ldots\right].$$

The absolute values of the coefficients

$$\frac{4}{\pi}, 0, \frac{-4}{3^2\pi}, 0, \frac{4}{5^2\pi}, \ldots, \frac{(\pm 1)4}{(2m-1)^2\pi}, \ldots$$

are respectively less than

$$\frac{C}{1^2}, \frac{C}{2^2}, \frac{C}{3^2}, \frac{C}{4^2}, \frac{C}{5^2}, \ldots, \frac{C}{(2m-1)^2}, \ldots$$

where we may have

$$C = \frac{8}{\pi}.$$

4. If $f(t)$ and its derivatives up to and including the $(s-1)$th are continous engineering functions in the interval $-\pi < t < \pi$ and

$$f^{(r)}(-\pi + 0) = f^{(r)}(\pi - 0)[r = 0, 1, \ldots (s-1)],$$

and if the sth derivative is an engineering function in $-\pi < t < \pi$, then the coefficients in the Fourier expansion of $f(t)$ are less in absolute value than $\frac{C}{n^{s+1}}$, where C is some positive number independent of n. To illustrate, let

$$f(t) = \frac{1}{48}\left(t^4 - 2\pi^2 t^2 + \frac{7\pi^4}{15}\right), \quad -\pi \leqq t \leqq \pi.$$

Then

$$f'(t) = \tfrac{1}{12}(t^3 - \pi^2 t),$$

$$f''(t) = \tfrac{1}{12}(3t^2 - \pi^2),$$

$$f^{(3)}(t) = \frac{t}{2}.$$

Now $f(t)$ and its first three derivatives are engineering functions, and

$$f^{(r)}(-\pi + 0) = f^{(r)}(\pi - 0). \quad (r = 0, 1, 2.),$$

$$f^{(3)}(-\pi + 0) \neq f^{(3)}(\pi - 0).$$

Thus in this case $s - 1 = 2$, or $s + 1 = 4$. The expansion of $f(t)$ is found to be

$$f(t) = \cos t - \frac{1}{2^4}\cos 2t + \frac{1}{3^4}\cos 3t - \ldots.$$

The absolute values of $\frac{1}{1^4}, -\frac{1}{2^4}, \frac{1}{3^4}, \ldots$, are respectively less than $\frac{C}{1^4}, \frac{C}{2^4}, \frac{C}{3^4}, \ldots, \frac{C}{n^4}, \ldots$, where C may be 2.

(*b*) *Differentiation and integration of Fourier series.* Sufficient conditions for the legitimacy of differentiating and integrating Fourier series are the following. If an engineering function $f(t)$ is continuous for all values of t in the interval $-\pi \leqq t \leqq \pi$ and if $f(-\pi + 0) = f(\pi - 0)$, then the first derivative of the function is equal to the derivative of its Fourier expansion.

If $f(t)$ is an engineering function, then the integral of $f(t)$ is equal to the integral of the Fourier expansion of $f(t)$.

(*c*) *Manipulation of Fourier series.* It was stated in § 35 that the Fourier expansion of an engineering function $f(t)$ in the interval $-\pi < t < \pi$ is unique. It can also be shown that the Fourier expansion of § 34 for the interval $0 < t < 2\pi$ is unique. The theorems of this section have been stated for the interval $-\pi$ to π. The Fourier development for this interval is frequently called *the* Fourier expansion, and all other Fourier expansions are viewed as special cases of this one. If the theorems of this section are desired for the interval 0 to 2π they are easily obtained by a linear change of independent variable $\tau = t + \pi$. Then $\tau = 0$ for $t = -\pi$ and $\tau = 2\pi$ for $t = \pi$. Or we may leave the theorems as they are and change the function. That is, if information is desired regarding the convergence of $f(t)$ in the interval $(0, 2\pi)$ let $\tau = t - \pi$ and expand $f(\tau) = f(t - \pi)$ by the formulas of § 33 for the interval $-\pi < \tau < \pi$. Then apply the theorems as stated.

In the derivation of formulas (142) and (143), it has been assumed that the series (141) possesses convergence of a nature (uniform convergence) that will permit the term by term integration there performed. (See Ref. 21 at the end of the text.)

40. Summary.

(a) To obtain the Fourier expansion of an engineering function defined in the interval $(-\pi, \pi)$ use Eqs. (142), (143), and (144). If the function is discontinuous (say at a point $t = a$) it is recalled from the calculus that

$$\int_{-\pi}^{\pi} \phi(t)dt = \int_{-\pi}^{a} \phi(t)dt + \int_{a}^{\pi} \phi(t)dt.$$

(b) If the interval of expansion is $(0, 2\pi)$ or $(0, \pi)$ or $(-l, l)$ or $(0, l)$, then the coefficients are given respectively by Eqs. (145), or (146) and (147), or (148), or (149) and (150).

(c) Harmonic analyses may be performed by the method of § 37.

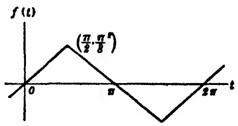
(d) Questions of convergence of Fourier expansions of engineering functions can be answered, in most cases, by the theorems of § 39.

EXERCISES AND PROBLEMS

1. In case 3, § 37, show that the formulas for a'_n and b'_n, when the interval is 0 to π, are the formulas already derived for the interval 0 to 2π.

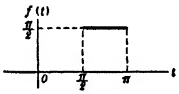
2. Obtain, for subsequent use, the Fourier expansions of the following functions.

(a)



Ans. $f(t) = \sin t - \frac{1}{3^2}\sin 3t + \frac{1}{5^2}\sin 5t - \dots.$

(b)



$$\textit{Ans. } f(t) = \sin t - \sin 2t + \tfrac{1}{3}\sin 3t + \tfrac{1}{5}\sin 5t - \tfrac{1}{3}\sin 6t + \tfrac{1}{7}\sin 7t + \dots.$$

or

$$f(t) = \frac{\pi}{4} - \cos t + \frac{1}{3}\cos 3t - \frac{1}{5}\cos 5t + \dots.$$

(c) $f(t) = e^t, \quad 0 < t < 2\pi.$

$$e^t = \frac{e^{2\pi} - 1}{\pi}\left[\frac{1}{2} + \frac{1}{1^2+1}\cos t + \frac{1}{2^2+1}\cos 2t + \frac{1}{3^2+1}\cos 3t + \dots - \frac{1}{1^2+1}\sin t - \frac{2}{2^2+1}\sin 2t - \frac{3}{3^2+1}\sin 3t - \dots\right].$$

(The derivative of e^t is not equal to the derivative of the series. In fact, the derivative of the series does not converge.)

(d) $f(t) = t^2. \quad -\pi < t < \pi.$

$$\textit{Ans. } f(t) = \frac{\pi^2}{3} - 4\left[\cos t - \frac{1}{2^2}\cos 2t + \frac{1}{3^2}\cos 3t - \dots\right].$$

Is the derivative of t^2 equal to the derivative of the series?

3. Given that $e^{int} = \cos nt + i \sin nt$, show that the series (141) and formulas (145) are reducible to the forms:

$$f(t) = \sum_{n=-\infty}^{n=\infty} \alpha_n e^{int},$$

where

$$2\alpha_n = a_n - ib_n, \quad n > 0$$
$$2\alpha_n = a_{-n} + ib_{-n}, \; n < 0$$
$$2\alpha_0 = a_0$$

and thus

$$\alpha_n = \frac{1}{2\pi}\int_0^{2} f(t)e^{-int}dt, \; (n = 0, \pm 1, \pm 2, \ldots).$$

This is the complex form for Fourier series. It will be used later in the text.

4. Complete the problem described in § 32. A synchronous machine is driven by a reciprocating engine, which supplies a torque that fluctuates periodically about an average value. It is desired to determine the angular position of the rotor of the synchronous machine as a function of time. The data are as follows:

$$T_s = 240{,}000 \text{ lb-ft. per radian,}$$
$$T_d = 1800 \text{ lb-ft. per radian per sec.,}$$
$$I = \frac{WR^2}{32.2}, \text{ where } WR^2 = 16{,}500 \text{ lb-ft.}^2,$$
$$\omega = 26.9 \text{ radians per sec.}$$

The engine torque may be found from the crank-effort curve, shown in Fig. 21. The crank effort is the tangential component of the total force transmitted to the crankpin

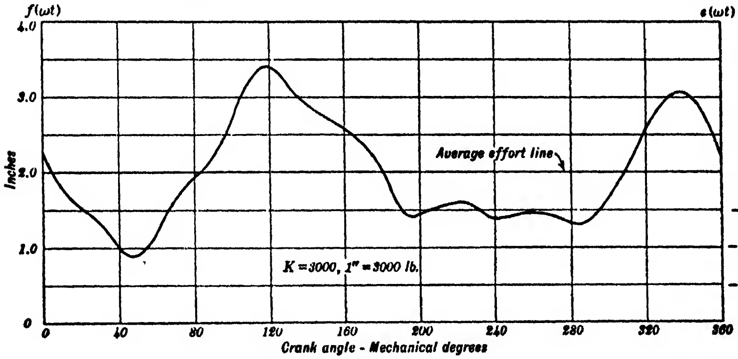


Fig. 21.—Crank-effort Curve.

by the connecting-rod. Thus, if the crank effort is denoted by e, and the radius of the crankpin by r, the engine torque $f(t)$ is given by

$$f(t) = re.$$

The crank effort as a function of crank angle θ is found from Fig. 21 by multiplying the ordinates y of the curve by a scale factor of 3000. These ordinates at 15° intervals are

ωt	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°
y	2.23	1.63	1.35	0.89	1.21	1.83	2.20	3.05	3.37	2.99	2.74	2.48
ωt	180°	195°	210°	225°	240°	255°	270°	285°	300°	315°	330°	345°
y	2.05	1.41	1.55	1.59	1.39	1.48	1.44	1.32	1.71	2.31	2.99	2.99

The radius of the crankpin is 1 ft.

First show by a Fourier analysis that the engine torque is approximately

$$\begin{aligned} f(t) = 6000 &+ 930 \sin \omega t - 330 \cos \omega t \\ &- 2310 \sin 2\omega t + 462 \cos 2\omega t \\ &- 720 \sin 3\omega t + 1110 \cos 3\omega t \\ &- 360 \sin 4\omega t + 30 \cos 4\omega t + \dots . \end{aligned}$$

The differential equation of motion is then

$$I\frac{d^2\theta}{dt^2} + Td\frac{d\theta}{dt} + T_s\theta = T_d\omega + T_s\omega t + 6000 + 930 \sin \omega t - 300 \cos \omega t - \dots .$$

(For the exact differential equation see Vol. II, Chap. I.) We are not primarily interested in the complementary function because the transient disturbance it represents soon dies away. Find the instantaneous displacement θ given by the particular integral. The periodic portion of the displacement is due to the periodic portion of the load. What is the maximum of the periodic displacement?

Show that if the flywheel moment of interia is improperly chosen, the system may resonate with one of the harmonics of the engine torque.

5. Suppose that the curve of Fig. 20 represents a voltage. Design a linear circuit such that the current due to this voltage will have a third harmonic component twice as great as the fundamental.

IV

SOLUTION OF HIGHER DEGREE AND TRANSCENDENTAL EQUATIONS

The purpose of Sec. IV is to present suitable methods, from an engineering standpoint, of solving numerical higher-degree algebraic and transcendental equations. The need for the solution of such equations has been partially indicated in the first part of the present chapter. The characteristic equations of simultaneous differential equations are likely to be of at least the fourth degree. Moreover, in vibrating mechanical and oscillating electrical systems the roots of the corresponding characteristic equations are generally complex quantities. Hence, it is frequently necessary to obtain all the complex roots of fourth-, sixth-, and higher-degree algebraic equations. Further need for the roots of such equations will arise in the study of Heaviside's operational calculus.

We desire solutions without a burden of formulas. Newton's method is employed for the solution of transcendental equations and Graeffe's method for the solution of higher-degree algebraic equations. The real roots of both transcendental and algebraic equations can be found by the method of successive approximations. This method has the merit that it rests on very little theory. It has the disadvantage that it is long and tedious if the equation is complicated.

41. Nature of Solutions of Algebraic Equations. Before developing these methods, the following facts should be noted:

(*a*) Algebraic formulas exist for the solution of the general quadratic, cubic, and quartic equations with literal (letter) coefficients. (See Ref. 24 at the end of the text.)

(*b*) No formulas exist for the solution of a general algebraic equation with literal coefficients if it is of higher degree than the fourth.

(*c*) Any rational integral equation $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$, whose coefficients are real or complex numbers has exactly n roots. These roots may be either real or complex.

(*d*) An equation $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$, where n is a positive odd integer and the coefficients in the equation are real, always has one real root.

(*e*) The number of positive roots of $a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$ (a's real) is either equal to the number of variations of signs of the a's or in less than this number of variations by an even integer (Descartes' rule of signs).

42. Newton's Method. Newton's method of solving equations is of engineering value because it not only gives the real roots of algebraic equations, but with equal ease it yields the real roots of tanscendental equations. Moreover, it is unnecessary to remember any formulas since those required are readily derived from the principles of the differential calculus. The development of Newton's method is as follows.

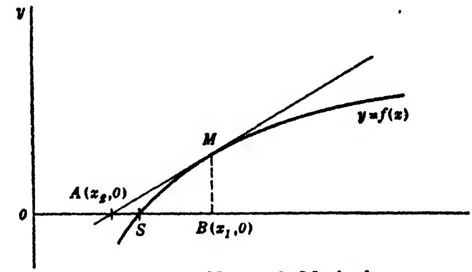


FIG. 22.—Newton's Method.

Suppose it is desired to find a solution of the equation $f(x) = 0$, when an approximate solution is known, as by estimate or from a rough graph. In Fig. 22 $y = f(x)$ is shown and OS is the root sought. Let $OB = x_1$ be the approximate solution, AM is the tangent to $y = f(x)$ at the point where $x = x_1$. (M must be sufficiently near to S that[12] $f'(x) \neq 0$ for any value of x between S and B.)

[12] The derivative of a function with respect to its argument is denoted by a prime thus

$$f'(x) = \frac{d}{dx} f(x)$$

and $f'(x_1)$ is the value of the derivative when $x = x_1$.

From the figure and the differential calculus

$$AB = \frac{f(x_1)}{f'(x_1)}.$$

If B is near S,

$$x_2 = OB - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{f(x_1)}{f'(x_1)}$$

is a closer approximation than x_1 to the root S of $f(x) = 0$. If next x_2 is used as an approximate root, in the same manner that x_1 has been employed, a third approximation to S is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

If this process is continued either the root of $f(x) = 0$, or a close approximation to it, is obtained.

EXAMPLE 1. Find the real root of the transcendental equation

$$f(x) = x - \tfrac{1}{2}\sin x - 1 = 0.$$

Let us first find an approximate solution x_1, by graphical means. Since this equation is the difference of two functions, it can be written in the form

$$\sin x = 2(x - 1).$$

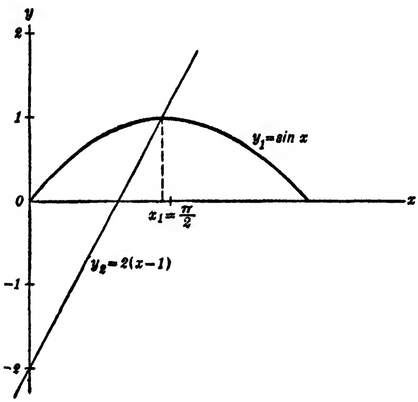


FIG. 23.

The abscissa of the point of intersection of the graphs of the two equations

$$y_1 = \sin x$$
$$y_2 = 2(x - 1)$$

is evidently a root of

$$x - \tfrac{1}{2}\sin x - 1 = 0.$$

The values of y_1 and y_2 are plotted in Fig. 23, from which an approximate value x_1 of the root is found to be

$$x_1 = \frac{\pi}{2}.$$

Then, applying Newton's method,

$$x_2 \quad x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{\pi}{\hat{}} - \frac{\left(\frac{\pi}{2} - \frac{1}{2}\sin\frac{\pi}{2} - 1\right)}{1 - \frac{1}{2}\cos\frac{\pi}{2}} = 1.5$$

and

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.5 - \frac{[1.5 - \frac{1}{2}\sin(1.5 \text{ radians}) - 1]}{1 - \frac{1}{9}\cos 1.5} = 1.498062.$$

Since $f(1.498062) = -\ 0.000616$ the process need not be repeated again for x_3 is a good approximation to the root.

EXAMPLE 2. Solve the equation $x^3 - 2x - 1 = 0$. Since

$$f(1) = -\ 2$$

and

$$f(2) = +\ 3,$$

a root must lie between $x = 1$ and $x = 2$. Take as an approximation

$$x_1 = 1.6.$$

Then

$$x_2 = 1.6 - \frac{[(1.6)^3 - 2(1.6) - 1]}{3(1.6)^2 - 2} = 1.618.$$

Since

$$f(1.618) = -\ 0.0002,$$

the value x_2 is a fair approximation to the root desired. Since $x = 1.618$ is an approximation root of $x^3 - 2x - 1$, it follows from the factor theorem that $x - 1.618$ is an approximate factor. Hence, upon division of $f(x)$ by $(x - 1.618)$ it is found that

$$x^3 - 2x - 1 = (x - 1.618)(x^2 + 1.618\,x + 0.618).$$

The other roots, which are roots of the quadratic factor, are obtained at once.

43. Successive Approximations. The method of successive approximations is based on the principle that a smooth curve, for a small interval of the independent variable, is almost a straight line within that interval. An example will make the method clear. The value $x = 1.6$ is an approximation to a root of

$$x^3 - 2x - 1 = 0.$$

Since

$$f(1.6) = -0.104,$$
$$f(1.7) = 0.513,$$

the graph of

$$y = x^3 - 2x - 1$$

crosses the x-axis between $x = 1.6$ and $x = 1.7$. If the graph is assumed to be a straight line through the points (1.6, − 0.104) and (1.7, 0.513), we have the relations shown in Fig. 24. From the figure

$$\frac{x_2}{0.1} = \frac{0.104}{0.617},$$

or

$$x_2 = 0.0168.$$

Hence a more accurate value of the solution is

$$x = 1.6 + x_2 = 1.6168.$$

By continuing the process, a closer approximation can be obtained.

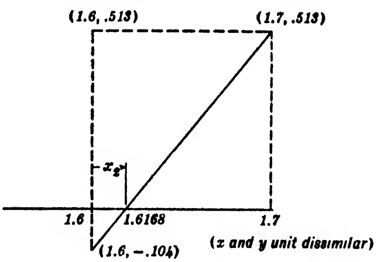


FIG. 24.

Newton's method and the method of successive approximation yield directly only the real roots. But the complex roots are the important ones in obtaining the periods of vibrations and oscillations in many electrical and mechanical problems. Since Graeffe's method gives all the roots of any algebraic equation, it is of great value.

44. Underlying Principle of Graeffe's Root-squaring Method. The underlying principle of Graeffe's method is readily explained by means of a quadratic equation whose roots are real and distinct. Consider the equation

$$x^2 + 10.1x + 1 = 0.$$

Suppose its roots ($x_2 = -10$, $x_1 = -\frac{1}{10}$) are unknown. There exists a simple routine method whereby we can transform

$$x^2 + 10.1x + 1 = 0$$

into an equation whose roots will be some high even power (say the 256th) of the roots of $x^2 + 10.1x + 1 = 0$. (It is, of course, not

necessary to know the roots of the original equation to do this.) The roots of the derived equation are then easily found, as may be seen by the following example. Let the derived equation be

$$x^2 - a_1x + a_2 = 0,$$

where a_1 and a_2 are known by the above-mentioned routine process which will be described later. Since the roots of the last equation are x_1^{256} and x_2^{256}, we have

$$x^2 - a_1x + a_2 = (x - x_1^{256})(x - x_2^{256})$$

$$= x^2 - (x_1^{256} + x_2^{256})x + (x_1x_2)^{256} = 0.$$

Hence

$$x_2^{256} + x_2^{256} = a_1 \tag{151}$$

and

$$(x_1x_2)^{256} = a_2. \tag{152}$$

Let x_2 designate the root with larger absolute value. Since $|x_2|>|x_1|$ we can neglect x_1^{256} in Eq. (151) and solve for x_2. (That is, in the solution of (151), $(-\frac{1}{10})^{256}$ is surely negligible in comparison with $(-10)^{256}$.) Since x_2 is now known, x_1 is easily determined from (152).

If the roots of the original equation are not widely separated, it may be necessary for the roots of the derived equation to be those of the original raised to a still higher power.

45. Preliminary Examples. We lead up to the general theory of Graeffe's method by the explanation of simple examples illustrating the different kinds of roots. The general theory will then be little more than a generalization of the notation used in the examples. Finally, from the general theory, a set of rules for the application of Graeffe's method will be derived.

EXAMPLE 1. Roots real and distinct. Let the roots of

$$x^2 + a_1x + a_2 = 0 \tag{153}$$

be $\mathbf{x}_1$ and $\mathbf{x}_2$, $|\mathbf{x}_2|>|\mathbf{x}_1|$. These roots with their signs changed are called the Encke roots of (153). Denote the Encke roots by x_1 and x_2. (The roots of the equation are denoted by bold-face type.) From the well-known relations between the roots and coefficients of a quadratic equation, we have

$$a_1 = -(\mathbf{x}_1 + \mathbf{x}_2) = x_1 + x_2,$$

$$a_2 = \mathbf{x}_1\mathbf{x}_2 = x_1x_2. \tag{154}$$

By means of these relations, it is evident that the equation whose Encke roots are the mth power of the roots of (153) is

$$x^2 + (x_1^m + x_2^m)x + x_1^m x_2^m = 0,$$

and also

$$x^2 + a_{11}x + a_{22} = 0, \tag{155}$$

where a_{11} and a_{22} are given by the routine process which will now be described.

Write (153) in the form

$$x^2 + a_2 = -a_1x.$$

Squaring and rearranging, we have

$$x^4 - (a_1^2 - 2a_2)x^2 + a_2^2 = 0.$$

If

$$x^2 = -y \tag{156}$$

then

$$y^2 + (a_1^2 - 2a_2)y + a_2^2 = 0. \tag{157}$$

By (156) the roots of (157) are the negative of the squares of (153), or the Encke roots of (157) are the squares of the Encke roots of (153). Applying, to (157), the process applied to (153), we have

$$y^4 - [(a_1^2 - 2a_2)^2 - 2a_2^2]y^2 + a_2^4 = 0. \tag{158}$$

If $y^2 = -z$, Eq. (158) is

$$z^2 + [(a_1^2 - 2a_2)^2 - 2a_2^2]z + a_2^4 = 0. \tag{159}$$

The Encke roots of (159) are the squares of the Encke roots of (157) and hence the fourth power of the Encke roots of (153). Eqs. (153), (157), and (159) may be respectively written

$$\begin{aligned} &x^2 + a_1x + a_2 = 0,\\ &y^2 + \begin{matrix} a_1^2 \\ -2a_2 \end{matrix}\; y + a_2^2 = 0,\\ &\quad\;\; + \begin{matrix} (a_1^2 - 2a_2)^2 \\ -\,2a_2^2 \end{matrix}\; z + a_2^4 = 0. \end{aligned} \tag{160}$$

By inspection, the following rule for writing a quadratic equation (called the **derived** equation), whose Encke roots are the squares of the Encke roots of a given equation, is seen to be as follows. The coefficient of any power of x in the derived equation is equal to the sum of the:

(a) Square of the coefficient of the corresponding power of x in the given equation plus

(b) The negative of twice the product formed by every pair of coefficients of powers of x which are equidistant from the power of x whose coefficient is being written.

If the squaring process of Eqs. (160) is continued, we obtain equations whose roots are the eighth, sixteenth, . . . and eventually the mth power of the roots of (153). Eq. (155) and the values of a_{11} and a_{22} are thus eventually obtained. But from

$$x_1^m + x_2^m = a_{11}$$

and

$$(x_1x_2)^m = a_{22}, \tag{161}$$

x_1 and x_2 can be found as explained in § 44.

Let it be required to solve, by the root-squaring method,

$$x^2 + 10.1x + 1 = 0.$$

The probability of error is diminished by a tabular arrangement of work.

TABLE I

	x^2 +	$10.1x$	+ 1
Squares of coefficients.......	1	102.01	1
Minus double products......	−	2	
Second-power roots.........	x^2 +	$100.01x$	+ 1
Square of coefficients	1	10002.0001	1
Minus double products......	−	2	
Fourth-power roots.........	x^2 +	$10000.0001x$	+ 1

By Eqs. (161), if x_1^4 is neglected, we have

$$x_2^4 = 10000.0001$$

$$x_2 = \pm\, 10.000,000,002,5-$$

$$x_1 = \pm\, 0.099,999,997,5+.$$

By checking in $x^2 + 10.1x + 1 = 0$, the algebraic signs of the roots are seen to be negative.

From an engineering point of view, the root-squaring process, in the numerical example just solved, was carried needlessly far. Consequently, a criterion is necessary for the termination of the root-

squaring process. To obtain such a criterion, for an equation with real and distinct roots, apply once more to Eq. (155) the rules following Eqs. (160) and obtain

$$x^2 + [(x_1^m + x_2^m)^2 - 2x_1^m x_2^m]x + x_1^{2m}x_2^{2m} = 0. \qquad (162)$$

The Encke roots of (162) are the $2m$th power of the Encke roots of (153). The Encke root x_2 of (153) was determined from (155) by the relation

$$x_2^m + x_1^m \doteq x_2^m = a_{11}.$$

Now, for m sufficiently large,

$$(x_2^m + x_1^m)^2 - 2x_1^m x_2^m,$$

in Eq. (162), is practically

$$(x_1^m + x_2^m)^2 \quad \text{or} \quad a_{11}^2.$$

Hence to compute x_2 of (153) from (162), we may write, for m sufficiently large,

$$(x_1^m + x_2^m)^2 \doteq x_2^{2m} = a_{11}^2,$$

or

$$x_2^m = \pm a_{11}.$$

But this is the same result as obtained from (155), and thus nothing was gained by an additional squaring.

Thus, it is evident that if the root-squaring process is carried far enough the coefficients of the next derived equation are practically the squares of the corresponding coefficients of the preceding equation. Accordingly, we have the RULE FOR THE TERMINATION OF THE ROOT-SQUARING PROCESS FOR DISTINCT REAL ROOTS:

(a) Determine beforehand the accuracy desired in the roots.

(b) Cease the process when the double-product term obtained in an additional squaring has no effect on the root to the accuracy desired. Let the roots of the numerical example be desired to two decimal places. It is evident from Table I that the double-product term, -2, does not have sufficient effect on the coefficient of x in the second derived equation,

$$x^2 + 10000.0001x + 1 = 0$$

to change the second decimal figure. Hence the last squaring was unessential.

EXAMPLE 2. Roots real and equal. Let the double root of

$$x^2 + 2a_1x + a_1^2 = 0 \qquad (163)$$

be x_1. Denote the Encke double root by x_1. A criterion, by the root-squaring method, for the existence of coincident roots of a quadratic equation is easily obtained as follows. The equation whose Encke roots are the mth power of the Encke roots of (163) is

$$x^2 + 2a_1^m x + a_1^{2m} = 0.$$

By the rules following Eqs. (160), which hold for all quadratic equations, the next derived equation is

$$x^2 + (4a_1^{2m} - 2a_1^{2m})x + a_1^{4m} = x^2 + 2a_1^{2m}x + a_1^{4m} = 0.$$

(When the roots were distinct, the process ultimately led to coefficients which were the squares of the coefficients of the preceding equation.) In this case, the coefficient of x in

$$x^2 + 2a_1^{2m}x + a_1^{4m} = 0$$

is only *half* the square of the coefficient of x in the preceding equation. Hence, the criterion for the quadratic equation is: If the coefficient of x_1, in successive equations formed by the root-squaring process, is only *half* the square of the coefficient of x in the preceding equation, then the original Eq. (163) has a double root.

The solution for the case of coincident roots is carried out exactly as for distinct roots. A table similar to Table I is constructed by the rules following Eqs. (160). Let the equation whose Encke roots are the mth power of the Encke roots of (163) be

$$x^2 + a_{11}x + a_{22} = 0.$$

Then

$$x_1^m + x_1^m = 2x_1^m = a_{11}$$

and

$$x_1^{2m} = a_{22}.$$

EXAMPLE 3. Roots complex. Let the Encke roots of

$$x^2 + a_1x + a_2 = 0 \tag{164}$$

be $re^{i\theta}$ and $re^{-i\theta}$. We desire a criterion, by the root-squaring process, for the existence of a pair of complex roots. The equation whose Encke roots are the mth power of the Encke roots of (164) is

$$(x + r^m e^{im\theta})(x + r^m e^{-im\theta}) = x^2 + 2r^m \cos m\theta + r^{2m} = 0. \tag{165}$$

If the root-squaring process is applied to the last equation, we have

$$x^2 + (4r^{2m}\cos^2 m\theta - 2r^{2m})x + r^{4m} = x^2 + 2r^{2m}\cos 2m\theta x + r^{4m} = 0. \tag{166}$$

The criterion for a pair of complex roots now consists of two observations:

(*a*) Since the angle θ in (166) is doubled by each squaring of the roots, the coefficient of x in (166) will frequently fluctuate in sign if m is sufficiently large.

(*b*) It is evident that in the coefficient, $(4r^{2m}\cos^2 m\theta - 2r^{2m})$, of x in (166) the double-product term $-2r^{2m}$ does not vanish in comparison with the squared coefficient $4r^{2m}\cos^2 m\theta$ as m increases. Thus with complex roots the root-squaring process is not continued (as was the case with real roots) until the double-product term becomes negligible compared to the square of the corresponding coefficient of the preceding equation. The point at which the process is stopped is made clear in the following example.

Let the roots of

$$x^2 - 2x + 2 = 0$$

be found by Graeffe's method.

TABLE II

	x^2	$-2x$	$+2$
	1	4	4
		-4	
Second power...........	1	0	4
	1	0	16
		-8	
Fourth power...........	1	-8	16

The equation whose Encke roots are the mth power of the roots of

$$x^2 - 2x + 2 = 0$$

is

$$x^2 + 2r^m\cos m\theta + r^{2m} = x^2 + a_{11}x + a_{22} = 0.$$

If we stop with $m = 2$ in Table II, we have

$$r^{2m} = r^4 = a_{22} = 4$$

or

$$r = \pm\sqrt{2}.$$

Let the Encke roots of

$$x^2 - 2x + 2 = 0$$

also be written

$$u \pm iv.$$

From the relations between roots and coefficients of a quadratic equation

$$u = 1.$$

Since

$$r = \sqrt{u^2 + v^2}, \quad v = \pm 1.$$

Thus the roots at this point are either $\pm (1 \pm i)$. By checking in in the original equation, the roots are seen to be $1 \pm i$.

If the process had been carried to $m = 4$, we have from Table II

$$r^{2m} = r^8 = a_{22} = 16$$

or

$$= \pm\sqrt{2}.$$

Thus the process was needlessly carried through the fourth power of the roots (in this case to illustrate the behavior of the coefficient of x).

With the insight thus acquired into the behavior of Graeffe's method as applied to simple examples, we consider his method as applied to the equation of the nth degree.

46. Graeffe's General Theory. By means of the introductory examples of § 45, the illustrative examples of this section, and the rules stated in § 47, any algebraic equation can be solved with very little recourse to the Graeffe's general theory. However, occasionally, questions arise which may not be taken care of by the rules. In this event, it is necessary to understand, and at times even to make slight extensions of, the general theory.

Consider the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0. \tag{167}$$

This may be written

$$x^n + a_2x^{n-2} + a_4x^{n-4} + \ldots = -(a_1x^{n-1} + a_3x^{n-3} + a_5x^{n-5} + \ldots).$$

Squaring both sides and rearranging, we have

$$x^{2n} - (a_1^2 - 2a_2)x^{2n-2} + (a_2^2 - 2a_1a_3 + 2a_4)x^{2n-4}$$
$$- (a_3^2 - 2a_2a_4 + 2a_1a_5 - 2a_6)x^{2n-6} + \ldots = 0. \tag{168}$$

Let the roots of (167), arranged in the order of their ascending *absolute* values, be

$$x_1, x_2, x_3, \ldots, x_n.$$

These numbers with their signs changed are the **Encke** roots of (167). Denote them by

$$x_1, x_2, \ldots, x_n.$$

If $x^2 = -y$, Eq. (168) may be written

$$y^n + \begin{bmatrix} +a_1^2 \\ -2a_2 \end{bmatrix} y^{n-1} + \begin{bmatrix} +a_2^2 \\ -2a_1a_3 \\ +2a_4 \end{bmatrix} y^{n-2} + \begin{bmatrix} +a_3^2 \\ -2a_2a_4 \\ +2a_1a_5 \\ -2a_6 \end{bmatrix} y^{n-3} + \ldots = 0. \tag{169}$$

The Encke roots of (169) are the squares of the Encke roots of (167).

By inspection of Eqs. (167–169), the law of formation of the coefficients of (169) may be stated thus: **The coefficient of any power of y in (169) is found by adding to the square of the corresponding coefficient in (167) twice the product of every pair of coefficients in (167) which are equally distant from the term considered, these products being taken with alternately negative and positive signs. An absent power of x is taken with the coefficient zero.**

This process may be repeated indefinitely, the resulting equations having Encke roots which are respectively the square, fourth power, eighth power, etc., of the Encke roots of (167). This process will now be applied to the solution of equations.

Case 1. All roots real and distinct. Let the Encke roots of (167) be $x_1, x_2, x_3, \ldots$ (The larger the subscript, the larger is the absolute value of the Encke root.) The well-known relations between the coefficients and the roots of an nth-degree equation are:

$$a_1 = x_n + x_{n-1} + \ldots + x_3 + x_2 + x_1 = [x_n]$$

$$a_2 = x_nx_{n-1} + x_nx_{n-2} + \ldots + x_1x_2 = [x_nx_{n-1}]$$

$$a_3 = x_nx_{n-1}x_{n-2} + x_nx_{n-1}x_{n-3} + \ldots + x_1x_2x_3 = [x_nx_{n-1}x_{n-2}]$$

The notation on the right is defined by these equations.

Making use of the last relations, we see that the equation whose Encke roots are the mth powers of those of (167) is

$$x^n + [x_n{}^m]x^{n-1} + [x_n{}^m x_{n-1}{}^m]x^{n-2} + \ldots = 0. \tag{170}$$

Now since

$$|x_n| > |x_{n-1}| > |x_{n-2}| \ldots,$$

it follows that $x_n{}^m$ is enormously greater than $x_{n-1}{}^m$ or $x_{n-2}{}^m$ for m sufficiently large. Thus the sum $[x_n{}^m]$ is very nearly equal to its first term $x_n{}^m$. In other words,

$$[x_n{}^m] = x_n{}^m(1 + d_1), \tag{171}$$

where d_1 is very small. Whence

$$\log |x_n| = \frac{1}{m}\log [x_n{}^m] - \frac{1}{m}\log (1 + d_1). \tag{172}$$

Now

$$[x_n{}^m], \quad [x_n{}^m x_{n-1}{}^m], \quad [x_n{}^m x_{n-1}{}^m x_{n-2}{}^m], \ldots$$

are known quantities obtained by the root-squaring process or from repeated applications of the rule following Eq. (169). If $\log (1 + d_1)$ is neglected in (172), the value of $|x_n|$ is obtained at once. Next,

$$[x_n{}^m x_{n-1}{}^m] = x_n{}^m x_{n-1}{}^m(1 + d_2), \tag{173}$$

where d_2 is very small. This equation is written

$$\log |x_{n-1}| = \frac{1}{m}\log [x_n{}^m x_{n-1}{}^m] - \frac{1}{m}\log [x_n{}^m] - \frac{1}{m}\log (1 + d_2). \tag{174}$$

If $\log (1 + d_2)$ is neglected, the value of $|x_{n-1}|$ is obtained. This process is continued until the absolute values of all the Encke roots are found. To obtain the roots of (167) it is necessary only to attach the proper algebraic sign to the absolute values determined by (171), (173) . . . . Whether the sign of x_n is positive or negative can be determined by the substitution of $\pm x_n$ in the original equation.

It is, of course, necessary to know when to cease increasing m. **The time to stop doubling m is when another doubling gives an equation whose roots are identical (to the number of figures originally decided upon) to those of the preceding equation.** The criterion for stopping the root-squaring process may be stated thus: **When the process yields coefficients in the next equation which are the squares of those of the preceding equation, to the accuracy required, the process is stopped.** This criterion is established as follows. Suppose the root-squaring process is applied to the equation whose Encke roots are already the mth power of the roots of the original equation. We then obtain, corresponding to Eq. (171), the relation

$$[x_n{}^{2m}] = x_n{}^{2m}(1 + \delta_1). \quad (\delta_1 \text{ small.})$$

But by the definition of $[x_n{}^{2m}]$ it is evident that, for $2m$ sufficiently large,

$$[x_n{}^{2m}] = [x_n{}^m]^2(1 + \delta_2). \quad (\delta_2 \text{ small.})$$

Replacing in this equation $[x_n{}^{2m}]$ by its equivalent value from the next to the last equation, we have

$$x_n{}^{2m}(1 + \delta_1) = [x_n{}^m]^2(1 + \delta_2).$$

If δ_1 and δ_2 are negligible,

$$x_n^m = [x_n^m].$$

But this is the value obtained from Eq. (171). Consequently, nothing is gained by an additional squaring of the roots when m has reached a value such that

$$[x_n^{2m}] = [x_n^m]^2(1 + \delta_2) \qquad (\delta_2 \text{ small}),$$

that is, when the process yields for the coefficient of x^{n-1} in the nex equation a value which is the square of the corresponding coefficient in the preceding equation, to the accuracy required. The reasoning is identical for the remaining Encke roots $x_{n-1}, \ldots, x_1$.

EXAMPLE 1. Solve

$$x^4 + 1.18x^3 - 1.856x^2 - 0.396x + 0.072 = 0.$$

The following tabulation, in view of the rule following Eq. (169), is self-explanatory.

TABLE III

	x^4	x^3	x^2	x	x^0
1st......	1	1.18	−1.856	−0.396	0.072
	1	1.3924 3.712	3.444736 0.93456 0.144	0.156816 0.267264	0.5184×10^{-2}
2nd.....	1	5.1044	4.523230	0.42408	0.5184×10^{-2}
	1	26.0549 −9.0466	20.4602 −4.3293 0.0104	0.179844 −0.046898	0.268739×10^{-4}
4th.....	1	17.0083	16.1413	0.132946	0.268739×10^{-4}
	1	289.2823 −32.2826	260.542 −4.522 0.000	1.76746×10^{-2} -0.08676×10^{-2}	7.22206×10^{-10}
8th.....	1	256.9997	256.020	1.6807×10^{-2}	7.22206×10^{-10}
	1	66,048.8 −512.0	65,546.2 −8.6	2.8248×10^{-4} -0.0037×10^{-4}	52.1582×10^{-20}
16th....	1	65,536.8	65,537.6	2.8211×10^{-4}	52.1582×10^{-20}

By Eqs. (172), (174), etc.,

$$\log |x_n| = \frac{1}{m}\log [x^m] = \frac{1}{16}\log 65536.8 = \frac{1}{16}(4.81649) = 0.30103.$$

$$x_n = \pm\, 2.000.$$

$$\log |x_{n-1}| = \frac{1}{m}(\log [x_n{}^m x_{n-1}{}^m] - \log [x_n{}^m]) = \frac{1}{16}(4.81649 - 4.81649)$$
$$= 0.0000.$$

$$x_{n-1} = \pm\, 1.000.$$

$$\log |x_{n-2}| = \frac{1}{m}(\log [x_n{}^m x_{n-1}{}^m x_{n-2}{}^m] - \log [x_n{}^m x_{n-1}{}^m])$$
$$= \tfrac{1}{16}(6.45042 - 10 - 4.81649) = 9.47712 - 10.$$

$$x_{n-2} = \pm\, 0.3000.$$

$$\log |x_{n-3}| = \frac{1}{m}(\log [x_n{}^m x_{n-1}{}^m x_{n-2}{}^m x_{n-3}{}^m] - \log [x_n{}^m x_{n-1}{}^m x_{n-2}{}^m])$$
$$= \tfrac{1}{16}[1.71732 - 20 - (6.45042 - 10)] = 9.07918 - 10.$$

$$x_{n-3} = \pm\, 0.1200.$$

Substituting in the original equation, the roots are $(-2, 1, -0.3, 0.12)$.

If Barlow's *Table of Squares, Cubes, etc.*, and Crelle's *Multiplication Tables* are available, the root-squaring tables are rapidly constructed and no questions of accuracy arise. If no tables or computing machines are available, the slide rule is, in general, sufficiently accurate for the construction of the root-squaring tables. This is due to the fact that the roots are largely determined by the exponent of 10 in the values of the coefficients of the mth-power equation. However, in the application of formulas (171), (173), . . . , logarithmic tables should be used which have one more place or decimal figure than the decimal place accuracy desired in the root. A summary of the procedure for all cases is given in § 47.

Case 2. All roots real, two or more being equal. For definiteness, let the Encke roots arranged in descending order of absolute magnitude be x_n, x_{n-1}, x_{n-2}, . . . x_1, with $x_{n-1} = x_{n-2}$. The equation corresponding to (170) is

$$x^n + [x_n{}^m]x^{n-1} + [x_n{}^m x_{n-1}{}^m]x^{n-2} + [x_n{}^m x_{n-1}{}^m x_{n-2}{}^m]x^{n-3} + \ldots = 0. \quad (175)$$

This equation, for m sufficiently large, is approximately

$$x^n + x_n{}^m x^{n-1} + 2x_n{}^m x_{n-1}{}^m x^{n-2} + x_n{}^m x_{n-1}^{2m} x^{n-3} + \ldots = 0. \quad (176)$$

Eq. (176) is called the **dominant** of (175) since only those terms in each coefficient are retained which greatly exceed, for m sufficiently large, the sum of all the others. (The dominant terms of $[x_n^m x_{n-1}^m]$ are $x_n^m x_{n-1}^m + x_n^m x_{n-2}^m = 2x_n^m x_{n-1}^m$.)

In the next equation of the root-squaring process, the coefficient of x^{n-2} will be

$$2x_n^{2m} x_{n-1}^{2m},$$

which is only half the square of the corresponding coefficient. (When the roots were distinct, the process ultimately led to coefficients which were the squares of the preceding ones.) Hence the rule: **If the coefficient of x^{n-s} in successive equations is only half the expected value, then the sth and $(s + 1)$th roots are equal.**

From the formation of the coefficients of (176), the solution for the repeated root is apparent. Evidently the $2m$th power of the repeated root is the quotient of the coefficients on either side of the irregular coefficient. (The coefficient of x^{n-3} is in the numerator.)

If two of the roots of an equation are equal in absolute value but opposite in sign, it is evident, from Eq. (172), that we still have case 2.

If three of the Encke roots are equal (say $x_{n-1} = x_{n-2} = x_{n-3}$), then the dominant equation is

$$x^n + x_n^m x^{n-1} + 3x_n^m x_{n-1}^m x^{n-2} + 3x_n^m x_{n-1}^{2m} x^{n-3} + \ldots = 0.$$

In the next derived equation the coefficient of x^{n-2} will be $3x_n^{2m} x_{n-1}^{2m}$ which is only one-third the expected value if all roots were distinct. Thus x_{n-1} may be computed by forming the quotient of the coefficients immediately following and immediately preceding the coefficient of x^{n-2}.

EXAMPLE 2. Solve

$$x^4 - 3.80x^3 + 3.17x^2 + 0.92\,x - 0.12 = 0.$$

Table IV is constructed by the rule following Eq. (169).

The next doubling of m would merely square each coefficient to four-place accuracy, except that of x^3. Now $512 = \frac{1}{2}\cdot 32^2$, whence the two largest roots are equal. These are

$$x_n = x_{n-1} = (6.553\cdot 10^4/1.000)^{1/16} = 2.000.$$
$$x_2 = (4.304/6.553\cdot 10^4)^{1/4} = 0.3000.$$
$$x_1 = (4.300\cdot 10^{-8}/4.304)^{1/4} = 0.1000.$$

There remains some question as to sign. By trial, it is found that the roots are 2, 2, -0.3, and 0.1.

TABLE IV

	x^4	x^3	x^2	x	x^0
1st	1	−3.80	3.17	0.92	−0.12
	1	14.44 −6.34	10.049 6.99 −0.24	0.8464 0.7608	0.0144
2nd	1	8.10	16.80	1.607	0.0144
	1	65.61 −33.60	282.24 −26.03 0.029	2.58 −0.48	2.074×10^{-4}
4th	1	32.01	256.24	2.10	2.074×10^{-4}
	1	1024.6 −512.5	65,659.0 −134.44 0.00	4.41 −0.106	4.3×10^{-8}
8th	1	512.1	65,525.0	4.304	4.3×10^{-8}

Case 3. One or more real roots and one pair of complex roots. For definiteness, let the Encke roots of (167), ($n = 4$) arranged in descending order be x_n, $re^{i\theta}$, $re^{-i\theta}$, x_{n-3}. The equation whose Encke roots are the mth power of these is

$$(x + x_n^m)(x + r^m e^{im\theta})(x + r^m e^{-im\theta})(x + x_{n-3}^m) = 0. \tag{177}$$

Performing the indicated multiplications, (177) becomes

$$\begin{aligned} x^4 &+ (x_n^m + x_{n-3}^m + 2r^m \cos m\theta)x^3 \\ &+ [r^{2m} + 2r^m(x_n^m + x_{n-3}^m) \cos m\theta + x_n^m x_{n-3}^m]x^2 \\ &+ [(x_n^m + x_{n-3}^m)r^{2m} + 2r^m x_n^m x_{n-3}^m \cos m\theta]x + x_n^m x_{n-3}^m r^{2m} = 0. \end{aligned} \tag{178}$$

When m is large, the last equation is approximately its dominant equation

$$x^4 + x_n^m x^3 + 2x_n^m r^m \cos m\theta x^2 + x_n^m r^{2m} x + x_n^m x_{n-3}^m r^{2m} = 0. \tag{179}$$

If the n Encke roots, in descending order (one pair complex), of an equation are

$$x_n, \quad re^{\pm i\theta} \quad x_{n-3}, \quad x_{n-4}, \quad \cdot\; x_1,$$

then its dominant equation is

$$x^n + x_n{}^m x^{n-1} + 2x_n{}^m r^m \cos m\theta \, x^{n-2} + x_n{}^m r^{2m} x^{n-3}$$
$$+ x_n{}^m r^{2m} x_{n-3}{}^m x^{n-4} + \ldots = 0. \quad (180)$$

As m is increased, the coefficient of x^{n-2} fluctuates in sign owing to the presence of the factor $\cos m\theta$. (The angle $m\theta$ is doubled at each application of the root-squaring process. See example 3, § 45. Hence if m is sufficiently large, $m\theta$ will not always lie in the first and fourth quadrants.) *This fluctuation in sign indicates a pair of complex roots.* Evidently the $2m$ power of the modulus r is the quotient of the coefficients on either side of the irregular coefficient. After r and x_n have been found, x_{n-3} can be found from the coefficient of x^{n-4}, and the remainder of the real roots from the coefficients of the lower powers of x.

To obtain the complex roots after their modulus has been obtained, proceed as follows. Let

$$re^{i\theta} = u + iv,$$

$$re^{-i\theta} = u - iv.$$

The sum of the Encke roots of the original equation then is

$$a_1 = x_n + 2u + x_{n-3} + x_{n-4} + \ldots + x_1,$$

or

$$u = \tfrac{1}{2}a_1 - \tfrac{1}{2}(x_n + x_{n-3} + x_{n-4} + \ldots + x_1).$$

The value of v is computed from

$$v = \quad r^2 - u^2$$

EXAMPLE 3. Solve the equation

$$f(x) = x^3 - 2x \quad 5 = 0.$$

Construct Table V by means of the rule following Eq. (169).

The fluctuation of the sign of the coefficient of x exhibits the presence of a pair of complex roots, the modulus of which is less than the real root. The real root is

$$(3.5467 \cdot 10^{20})^{1/64} = \pm\, 2.0944.$$

Now

$$f(2.0) = -\, 1.0$$

$$f(2.1) = 0.06$$

TABLE V

	x^3	x^2	x	x^0
1st.......	1	0.0	−2.0	−5.0
	1	0.0 4.0	4.0 0.0	25.0
2nd.....	1	4.0	4.0	25.0
	1	16.0 −8.0	16.0 −200.0	625.0
4th......	1	8.0	−184.0	625.0
	1	64.0 368.0	33,856.0 −10,000.0	390,625.0
8th.....	1	4.32×10^2	2.3856×10^4	3.90625×10^5
	1	1.86624×10^5 -0.47712×10^5	5.69109×10^8 $-3\ 37500 \times 10^8$	1.52588×10^{11}
16th.....	1	1.38912×10^5	2.31609×10^8	1.52588×10^{11}
	1	1.92965×10^{10} -0.04632×10^{10}	5.36427×10^{16} -4.23926×10^{16}	2.328309×10^{22}
32nd.....	1	1.88333×10^{10}	1.12501×10^{16}	2.32831×10^{22}
	1	$3\ 54693 \times 10^{20}$ -0.00025×10^{20}	1.2656×10^{32} -8.769952×10^{32}	5.4210×10^{44}
64th.....	1	3.54668×10^{20}	-7.5044×10^{32}	5.4210×10^{44}

whence there is a real root between $x = 2.0$ and $x = 2.1$. The sign of 2.0945 is therefore positive. Again,

$$r^2 = (5.421 \cdot 10^{44}/3.5467 \cdot 10^{20})^{1/64} = 2.387,$$

$$u = 0 + \tfrac{1}{2} \cdot 2.095 = 1.0472,$$

$$u^2 = 1.097,$$

$$v^2 = r^2 - u^2 = 2.387 - 1.097 = 1.290,$$

$$v = \pm 1.1357,$$

whence, finally,

$$x = 2.0944, \quad -1.0472 \pm i\,1.1357.$$

More accurate values are

$$x = 2.09455, \quad -1.04728 \pm i\,1.13594.$$

Case 3 is summarized in § 47.

Case 4. Four distinct complex roots. An equation of the fourth degree all of whose roots are, in general, complex may occur as the characteristic or auxiliary equation associated with the differential equations of an electric circuit such as Fig. 10, or of an oscillatory mechanical system possessing two degrees of freedom, as in problem 1, § 20.

Let the Encke roots of

$$x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$$

arranged in descending order be $r_2e^{\pm i\phi_2}$ and $r_1e^{\pm i\phi_1}$. The equation whose Encke roots are the mth power of these is

$$(x + r_2^me^{im\phi_2})(x + r_2^me^{-im\phi_2})(x + r_1^me^{im\phi_1})(x + r_1^me^{-im\phi_1}) = 0,$$

or

$$\begin{aligned} x^4 &+ 2(r_2^m \cos m\phi_2 + r_1^m \cos m\phi_1)x^3 \\ &+ (r_2^{2m} + 4r_2^mr_1^m \cos m\phi_2 \cos m\phi_1 + r_1^{2m})x^2 \\ &+ 2(r_2^mr_1^{2m} \cos m\phi_2 + r_1^mr_2^{2m} \cos m\phi_1)x + r_1^{2m}r_2^{2m} = 0. \end{aligned}$$

The dominant of the last equation is

$$x^4 + 2(r_2^m \cos m\phi_2)x^3 + r_2^{2m}x^2 + 2(r_1^mr_2^{2m} \cos m\phi_1)x + r_1^{2m}r_2^{2m} = 0. \qquad (181)$$

Let us examine the behavior of the coefficients of (181) when the root-squaring process is again applied. By the rule following Eq. (169), we have

$$\begin{aligned} x^4 &+ \begin{bmatrix} 4(r_2^m \cos m\phi_2)^2 \\ \quad - 2r_2^{2m} \end{bmatrix} x^3 \\ &+ \left. \begin{matrix} r_2^4 \\ - 8r_2^mr_1^m(r_2^m \cos m\phi_2)(r_2^m \cos m\phi_1) \\ + 2r_1^{2m}r_2^{2m} \end{matrix} \right] x^2 \\ &+ \begin{bmatrix} 4(r_1^mr_2^{2m} \cos m\phi_1)^2 \\ - 2r_1^{2m}r_2^{4m} \end{bmatrix} x \\ &+ (r_1r_2)^{4m} = 0. \end{aligned}$$

In the coefficients of x^3 and x the double-product terms do not necessarily vanish, in comparison with the square terms, as m is increased. These coefficients eventually fluctuate in sign. The coefficients of x^4, x^2, and x^0 are regular (i.e., the double-product terms vanish in comparison with the squared terms), and the root-squaring process is stopped by the application of the rule following Eq. (174) to the regular coefficients. The coefficient of x^2 in Eq. (181) is r_2^{2m}. After the value of r_2^2 is obtained, the constant term yields the value of r_1^2.

To obtain the real and imaginary parts of the Encke roots, let

$$r_2 e^{\pm i\phi_2} = u_2 \pm iv_2,$$

$$r_1 e^{\pm i\phi_1} = u_1 \pm iv_1.$$

The original equation then is

$$x^4 + 2(u_1 + u_2)x^3 + (u_1^2 + v_1^2 + 4u_1u_2 + u_2^2 + v_2^2)x^2$$
$$+ 2[u_2(u_1^2 + v_1^2) + u_1(u_2^2 + v_2^2)]x + (u_1^2 + v_1^2)(u_2^2 + v_2^2) = 0. \quad (182)$$

The relations between the Encke roots and coefficients give:

$$2(u_1 + u_2) = a_1, \quad (183)$$

$$2(u_2 r_1^2 + u_1 r_2^2) = a_3, \quad (184)$$

$$4u_1u_2 = a_2 - r_1^2 - r_2^2. \quad (185)$$

Finally, the values of v_1 and v_2 are given by the relations

$$v_1 = \pm\sqrt{r_1^2 - u_1^2},$$

$$v_2 = \pm\sqrt{r_2^2 - u_2^2}.$$

The algebraic signs of all the roots are determined by substitution in the original equation.

EXAMPLE 4. Let us obtain, to one decimal place, the roots of

$$x^4 + 25.5x^3 + 685x^2 + 8150x + 47{,}400 = 0.$$

We construct Table VI by means of the rule following Eq. (169). Where the double-product terms may be neglected, an asterisk (*) appears.

The fluctuation of the signs of the coefficients of x^3 and x exhibits the presence of two pairs of complex roots. From the dominant equation and Table VI

$$(r_2^2)^{32} = 1.608 \times 10^{84}$$

TABLE VI

	x^4	x^3	x^2	x	x^0
1st......	1	2.55×10	6.85×10^2	8.15×10^3	4.74×10^4
	1	6.5025×10^2 -13.70×10^2	4.6922×10^5 -4.156×10^5 9.48×10^4	6.6422×10^7 -6.4938×10^7	2.2468×10^9
2nd.....	1	-7.198×10^2	1.484×10^5	1.485×10^6	2.2468×10^9
	1	5.1811×10^5 -2.968×10^5	2.2023×10^{10} 2.136×10^9 4.494×10^9	2.2023×10^{12} -6.6685×10^{14}	5.0481×10^{18}
4th.....	1	2.213×10^5	2.8653×10^{10}	-6.646×10^{14}	5.0481×10^{18}
	1	4.897×10^{10} -5.731×10^{10}	8.2099×10^{20} 2.942×10^{20} 1.010×10^{19}	4.417×10^{29} -2.893×10^{29}	2.548×10^{37}
8th.....	1	-8.34×10^9	1.1253×10^{21}	1.524×10^{29}	2.548×10^{37}
		6.956×10^{19} -2.251×10^{21}	1.266×10^{42} 2.542×10^{39} 5.096×10^{37}	2.323×10^{58} -5.735×10^{58}	6.492×10^{74}
16th....	1	-2.181×10^{21}	1.268×10^{42}	-3.412×10^{58}	6.492×10^{74}
	1	4.757×10^{42} -2.536×10^{42}	1.608×10^{84} * *	1.164×10^{117} -1.646×10^{117}	4.215×10^{149}
32nd....	1	2.221×10^{42}	1.608×10^{84}	-4.82×10^{116}	4.215×10^{149}

or

$$r_2 = 428.0$$

$$(r_1^2 r_2^2)^{32} = 4.215 \times 10^{149}$$

or

$$r_1^2 = 110.74.$$

Eqs. (183) and (184) give

$$2(u_1 + u_2) = 25.5$$

$$2(u_2 r_1^2 + u_1 r_2^2) = 8150,$$

whence
$$u_1 = 8.39$$
$$u_2 = 4.36$$
and
$$v_1 = \pm\sqrt{r_1^2 - u_1^2} = \pm 6.35$$
$$v_2 = \pm\sqrt{r_2^2 - u_2^2} = \pm 20.20.$$

The roots are found to be

$$-8.39 \pm 6.35i, \quad -4.36 \pm 20.2i.$$

Eq. (185) may be used as a check.

Case 5. Six distinct complex roots. An equation with six distinct complex roots arises in the study of the motion of a rigid body, such as a top, which is in rotation but one point of which is fixed. The orientation of the body about the fixed point at any time t may be specified by three angles, for example, the Eulerian angles.

The motion of the body following a displacement from an equilibrium configuration is described by three second-order differential equations, which are linear or frequently can be reduced to linear equations by justifiable approximations (such as were employed in problem 6, § 10). Then the characteristic equation of the differential equation system is of the sixth degree and, in general, due to the oscillatory character of the motion, has six distinct complex or imaginary roots.

The study of transient oscillations in linear circuits having three branches, each of which has inductance, capacitance, and resistance, will in general necessitate the solution of a sixth-degree equation having complex roots.

Although case 5 closely resembles case 4, yet because of its frequent occurrence a detailed study is made and an illustrative example solved. Let the Encke roots, arranged in descending order, of

$$x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6 = 0 \tag{186}$$

be $r_3e^{\pm i\phi_3}$, $r_2e^{\pm i\phi_2}$, and $r_1e^{\pm i\phi_1}$. The equation whose Encke roots are the mth power of the Encke roots of (186) is

$$(x + r_3{}^m e^{im\phi_3})(x + r_3{}^m e^{-im\phi_3})(x + r_2{}^m e^{im\phi_2})(x + r_2{}^m e^{-im\phi_2})$$
$$\cdot(x + r_1{}^m e^{im\phi_1})(x + r_1{}^m e^{-im\phi_1}) = 0,$$

or

$$x^6 + 2(r_3{}^m \cos m\phi_3 + r_2{}^m \cos m\phi_2 + r_1{}^m \cos m\phi_1)x^5$$
$$+ [(r_3{}^{2m} + r_2{}^{2m} + r_1{}^{2m}) + 4(r_3{}^m r_2{}^m \cos m\phi_3 \cos m\phi_2$$

$$+ r_3{}^m r_1{}^m \cos m\phi_3 \cos m\phi_1 + r_2{}^m r_1{}^m \cos m\phi_2 \cos m\phi_1)]x^4$$
$$+ 2[(r_3{}^m r_2{}^{2m} + r_3{}^m r_1{}^{2m}) \cos m\phi_3 + (r_2{}^m r_1{}^{2m} + r_2{}^m r_3{}^{2m}) \cos m\phi_2$$
$$+ (r_1{}^m r_2{}^{2m} + r_1{}^m r_3{}^{2m}) \cos m\phi_1 + 4r_1{}^m r_2{}^m r_3{}^m \cos m\phi_3 \cos m\phi_2 \cos m\phi_1]x^3$$
$$+ [(r_3{}^{2m} r_2{}^{2m} + r_3{}^{2m} r_1{}^{2m} + r_2{}^{2m} r_1{}^{2m}) + 4r_3{}^{2m} r_2{}^m r_1{}^m \cos m\phi_2 \cos m\phi_1$$
$$+ 4r_2{}^{2m} r_3{}^m r_1{}^m \cos m\phi_1 \cos m\phi_3 + 4r_1{}^{2m} r_3{}^m r_1{}^m \cos m\phi_2 \cos m\phi_3]x^2$$
$$+ 2(r_3{}^m r_1{}^{2m} r_2{}^{2m} \cos m\phi_3 + r_3{}^{2m} r_2{}^m r_1{}^{2m} \cos m\phi_2 + r_3{}^{2m} r_1{}^m r_2{}^{2m} \cos \phi_1)x$$
$$+ r_1{}^{2m} r_2{}^{2m} r_3{}^{2m} = 0. \tag{187}$$

The dominant equation of (187) is

$$x^6 + (r_3{}^m \cos m\phi_3)x^5 + r_3{}^{2m}x^4 + 2[(r_2{}^m r_3{}^{2m}) \cos m\phi_2]x^3 + r_3{}^{2m} r_2{}^{2m} x^2$$
$$+ 2[r_3{}^{2m} r_1{}^m r_2{}^{2m} \cos m\phi_1]x + r_1{}^{2m} r_2{}^{2m} r_3{}^{2m} = 0. \tag{188}$$

When the root-squaring process has been applied once more by the application of the rule following Eq. (169), the following facts are in evidence. In the coefficients of x^5, x^3, and x the double-product terms do not necessarily vanish, in comparison with the squared terms, as m is increased. These coefficients fluctuate in sign for m sufficiently large. The coefficients of x^6, x^4, x^2, and x^0 are regular and the root-squaring process is stopped by the application of the rule following Eq. (174) to the regular coefficients.

The coefficient of x^4 in Eq. (188) is $r_3{}^{2m}$. The quotient of the coefficients of x^2 and x^4 gives $r_2{}^{2m}$, and finally the quotient of the coefficients of x^0 and x^2 is $r_1{}^{2m}$.

To obtain the real and imaginary parts of the Encke roots, let

$$r_k e^{\pm i\phi_k} = u_k \pm iv_k, \qquad (k = 1,2,3).$$

The equation corresponding to (182) of case 4 is easily written down. After this equation has been written, a comparison of its coefficients with those of Eq. (186) gives the relations

$$2(u_1 + u_2 + u_3) = a_1, \tag{189}$$
$$r_1{}^2 + r_2{}^2 + 4u_1u_2 + 4(u_1 + u_2)u_3 + r_3{}^2 = a_2, \tag{190}$$
$$2[u_2r_1{}^2 + u_1r_2{}^2 + u_3(r_1{}^2 + 4u_1u_2 + r_2{}^2) + (u_1 + u_2)r_3{}^2] = a_3,$$
$$r_1{}^2r_2{}^2 + 4u_3(u_2r_1{}^2 + u_1r_2{}^2) + (r_1{}^2 + 4u_1u_2 + r_2{}^2)r_3{}^2 = a_4,$$
$$2u_3r_1{}^2r_2{}^2 + 2r_3{}^2(u_2r_1{}^2 + u_1r_2{}^2) = a_5. \tag{191}$$

Since $r_1{}^2$, $r_2{}^2$, and $r_3{}^2$ are known, the simultaneous solution of (189), (191), and (say, 190) gives one or more values for u_1, u_2, and u_3.

Double values for the variables may arise since one of the equations for the determination of the u's is quadratic. The two remaining equations of (189–191) may be used as a check to obtain the proper signs of the real parts of the roots and to reject the extraneous roots introduced by the quadratic equation.

EXAMPLE 5. Let us solve the equation

$$x^6 + 58x^5 + 1682x^4 + 11{,}648x^3 + 42{,}436x^2 + 61{,}800x + 45{,}000 \quad 0.$$

Table VII is constructed by the rule following Eq. (169).

Comparison of the values of the coefficients of x^4, x^2, and x^0, as given in Table VII, with their expressions in (188) gives respectively

$$(r_3^2)^{16} = 3.556 \times 10^{49},$$

$$(r_3^2r_2^2)^{16} = 4.275 \times 10^{69},$$

$$(r_3^2r_2^2r_1^2)^{16} = 2.829 \times 10^{74},$$

or

$$r_3^2 = 1250.06, \quad r_2^2 = 17.99, \; r_1^2 = 2.001.$$

By Eqs. (189–191),

$$u_3 = 25.00, \quad u_2 = 2.999, \quad u_1 = 1.002.$$

From

$$v_k = \sqrt{r_k^2 - u_k^2}, \qquad (k = 1,2,3)$$

we have

$$v_3 = \pm\, 25.00, \quad v_2 = \pm\, 2.999, \quad v_1 = \pm\, 1.002.$$

The values found for the roots (not Encke roots) are

$$x = -\,25 \pm i25, \quad -2.999 \pm i2.999, \quad -1.002 \pm i1.002.$$

The correct values are

$$x = -\,25 \pm i25, \quad -3 \pm i3, \quad -1 \pm i.$$

Case 6. Coincident complex roots. If two or more of the natural frequencies (§ 12) of oscillation of the body described in case 5 coincide, the characteristic equation will have coincident complex roots. The same statement is true for electrical circuits. Since resonance is so often either to be secured or avoided in engineering, but never ignored, this case also is important.

Let the Encke roots, arranged in descending order, of

$$x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = 0 \qquad (192)$$

TABLE VII

	x^6	x^5	x^4	x^3	x^2	x	x^0
1st.	1	5.8×10	1.682×10^3	1.1648×10^4	4.2436×10^4	6.18×10^4	4.5×10^4
	1	3.364×10^3 -3.364×10^3	2.829×10^6 -1.351×10^6 8.487×10^4	1.357×10^8 -1.428×10^8 7.169×10^6 -9.000×10^4	1.801×10^9 -1.440×10^9 1.514×10^8	3.819×10^9 -3.819×10^9	2.025×10^9
2nd.	1	0	1.563×10^6	0	5.12×10^8	0	2.025×10^9
	1	0 -3.126×10^6	2.443×10^{12} 0 1.024×10^9	0 -1.601×10^{15} 0 -4.050×10^9	2.621×10^{17} 0 6.330×10^{15}	0 -2.074×10^{18}	4.101×10^{18}
4th.	1	-3.126×10^6	2.444×10^{12}	-1.601×10^{15}	2.684×10^{17}	-2.074×10^{18}	4.101×10^{18}
	1	9.772×10^{12} -4.888×10^{12}	5.973×10^{24} -1.001×10^{22} *	2.563×10^{30} -1.312×10^{30} 1.297×10^{25} *	7.204×10^{34} -6.641×10^{33} 2.005×10^{31}	4.301×10^{36} -2.201×10^{36}	1.682×10^{37}
8th.	1	4.884×10^{12}	5.963×10^{24}	1.251×10^{30}	6.542×10^{34}	2.1×10^{36}	1.682×10^{37}
	1	2.385×10^{25} -1.193×10^{25}	3.556×10^{49} *	1.565×10^{60} -7.802×10^{59} *	$4\ 280 \times 10^{69}$ -5.254×10^{66} *	4.41×10^{72} -2.20×10^{72}	2.829×10^{74}
16th.	1	$1\ 192 \times 10^{25}$	3.556×10^{49}	$7\ 85 \times 10^{58}$	4.275×10^{69}	2.21×10^{72}	2.829×10^{74}

be x_1 and $re^{\pm i\phi}$ (double roots). The equation whose Encke roots are the mth power of the Encke roots of (192) is

$$(x + x_1^m)(x + r^m e^{im\phi})^2(x + r^m e^{-im\phi})^2 = 0,$$

or

$$\begin{aligned} x^5 &+ (x_1^m + 4r^m \cos m\phi)x^4 \\ &+ [4r^m x_1^m \cos m\phi + 2r^{2m}(2 + \cos 2m\phi)]x^3 \\ &+ [4r^{3m} \cos m\phi + 2r^{2m}(2 + \cos 2m\phi)x_1^m]x^2 \\ &+ (r^{4m} + 4r^{3m}x_1^m \cos m\phi)x + r^{4m}x_1^m = 0. \end{aligned} \tag{193}$$

The dominant equation of (193) is

$$\begin{aligned} x^5 &+ x_1^m x^4 + 4r^m x_1^m \cos m\phi x^3 \\ &+ 2r^{2m}(2 + \cos 2m\phi)x_1^m x^2 + 4r^{3m}x_1^m \cos m\phi x \\ &+ r^{4m}x_1^m = 0. \end{aligned} \tag{194}$$

(Coefficients in the dominant equation may consist of more than one term in case of multiple roots.)

If the root-squaring process is applied once more to (194), it is found that:

> The coefficients of x^5, x^4, and x^0 are regular while those of x^3, x^2, and x (adjacent terms) are irregular.
>
> The coefficients of x^3 and x fluctuate in sign for m sufficiently large. The coefficient of x^2 eventually becomes positive as m is increased. In the $2m$th equation, for m sufficiently large, the dominant part of the coefficient of x^2 is
>
> $$2x_1^{2m}r^{4m}(2 + \cos 4m\phi).$$
>
> Its expected value, if regular, would be the square of the coefficient of x^2 in Eq. (194), that is
>
> $$4x_1^{2m}r^{4m}(2 + \cos 2m\phi)^2.$$
>
> The root-squaring process is stopped by the application of the rule following Eq. (174) to the *regular* coefficients.

The value of x_1 is obtained from the coefficient of x^4, and in turn r^2 is obtained from the constant term of (194).

Next suppose $r > |x_1|$. Then the dominant equation of (193) is

$$\begin{aligned} x^5 &+ (4r^m \cos m\phi)x^4 + 2r^{2m}(2 + \cos 2m\phi)x^3 \\ &+ (4r^{3m} \cos m\phi)x^2 \\ &+ r^{4m}x + r^{4m}x_1^m = 0. \end{aligned} \tag{195}$$

In (195), the coefficients of x^5, x, and x^0 are regular while those of x^4, x^3, and x^2 are irregular. The values of r^2 and x_1 are given by the coefficients of the last two terms.

EXAMPLE 6. Let us solve the equation

$$x^5 + x^4 - 4x^3 - 16x^2 - 20x - 12 = 0.$$

Table VIII is constructed by the rule following Eq. (169).

Evidently in Table VIII the coefficients of x^5, x^4, and x^0 are regular while those of x^3, x^2, and x are irregular.

Comparison of the coefficients of x^4 and x^0 in Table VIII and Eq. (194) gives

$$x_1^{32} = 1.852 \times 10^{15},$$

$$x_1^{32}(r^4)^{32} = 3.4225 \times 10^{34},$$

or

$$x_1 = \pm\ 2.9999$$

and

$$r^2 = 2.000.$$

By the factor theorem, the root is $2.9999 \doteq 3$. The sum of the real parts of the Encke roots equals the coefficient of x^4 in the original equation. Consequently, if

$$re^{\pm i\phi} = u \pm iv,$$

then

$$4u - x_1 = 1 \qquad (x_1 \text{ is the Encke root}),$$

$$4u = 1 + 2.9999 = 3.9999 \doteq 4,$$

$$u = 1,$$

$$v = \pm \sqrt{2 - 1} = \pm\ 1.$$

The roots are 3, $1 \pm i$, $1 \pm i$.

Case 7. Seventh degree, two pairs of complex roots. As a final case, consider the seventh-degree equation

$$x^7 + a_1x^6 + a_2x^5 + a_3x^4 + a_4x^3 + a_5x^2 + a_6x + a_7 = 0, \quad (196)$$

whose Encke roots are x_3, x_2, x_1, $r_2e^{\pm i\phi_2}$, and $r_1e^{\pm i\phi_1}$. The equation whose Encke roots are the mth power of the Encke roots of (196) is

$$(x + x_3^m)(x + x_2^m)(x + x_1^m)(x + r_2^me^{im\phi_2})(x + r_2^me^{-im\phi_2}) \text{ times}$$
$$(x + r_1^me^{im\phi_1})(x + r_1^me^{-im\phi_1}) = 0,$$

or

$$x^7 + a'_1x^6 + a'_2x^5 + a'_3x^4 + a'_4x^3 + a'_5x^2 + a'_6x + a'_7 = 0, \quad (197)$$

TABLE VIII

	x^5	x^4	x^3	x^2	x	x^0
1st.	1	1.0	-4.0	-1.6×10	-2.0×10	-1.2×10
	1	1.0 8.0	1.6×10 3.2×10 $-4\ 0 \times 10$	2.56×10^2 -1.60×10^2 -2.4×10	4.0×10^2 -3.84×10^2	1.44×10^2
2nd.	1	9.0	8.0	7.2×10	1.6×10	1.44×10^2
	1	8.1×10 -1.6×10	6.4×10 -1.296×10^3 3.2×10	5.18×10^3 -2.56×10^2 2.59×10^3	2.56×10^2 -2.074×10^4	2.074×10^4
4th.	1	6.5×10	-1.2×10^3	7.51×10^3	-2.048×10^4	$2\ 074 \times 10^4$
	1	4.225×10^3 2.40×10^3	1.44×10^6 -9.76×10^5 -4.10×10^4	5.640×10^7 -4.915×10^7 2.696×10^6	4.194×10^8 -3.115×10^8	4.301×10^8
8th.	1	6.625×10^3	4.23×10^5	9.946×10^6	$+1.079 \times 10^8$	4.301×10^8
	1	4.389×10^7 -8.46×10^5	1.789×10^{11} -1.318×10^{11} $+2.158 \times 10^8$	9.892×10^{13} -9.128×10^{13} 5.699×10^{12}	1.164×10^{16} -8.556×10^{15}	1.850×10^{17}
16th.	1	4.304×10^7	4.73×10^{10}	1.334×10^{13}	3.08×10^{15}	1.850×10^{17}
	1	1.852×10^{15} -9.46×10^{10}	2.237×10^{21} -1.148×10^{21} 6.16×10^{15}	1.780×10^{26} -2.914×10^{26} 1.592×10^{25}	9.486×10^{30} -4.936×10^{30}	3.4225×10^{34}
32nd.	1	1.852×10^{15}	$+1.089 \times 10^{21}$	9.75×10^{25}	4.550×10^{30}	3.4225×10^{34}

where

$$a'_1 = (x_3^m + x_2^m + x_1^m) + 2(r_2^m \cos m\phi_2 + r_1^m \cos m\phi_1),$$

$$\begin{aligned} a'_2 = {} & x_3^m x_2^m + x_3^m x_1^m + x_2^m x_1^m \\ & + 2(x_3^m + x_2^m + x_1^m)(r_2^m \cos m\phi_2 + r_1^m \cos m\phi_1) \\ & + r_2^{2m} + r_1^{2m} + 4r_2^m r_1^m \cos m\phi_2 \cos m\phi_1, \end{aligned}$$

$$\begin{aligned} a'_3 = {} & x_3^m x_2^m x_1^m \\ & + 2(x_3^m x_2^m + x_3^m x_1^m + x_2^m x_1^m)(r_2^m \cos m\phi_2 + r_1^m \cos m\phi_1) \\ & + (x_3^m + x_2^m + x_1^m)(r_2^{2m} + r_1^{2m} + 4r_2^m r_1^m \cos m\phi_2 \cos m\phi_1) \\ & + 2r_2^m r_1^m (r_2^m \cos m\phi_1 + r_1^m \cos m\phi_2), \end{aligned}$$

$$\begin{aligned} a'_4 = {} & 2x_3^m x_2^m x_1^m (r_2^m \cos m\phi_2 + r_1^m \cos m\phi_1) \\ & + (x_3^m x_2^m + x_3^m x_1^m + x_2^m x_1^m)(r_2^{2m} + r_1^{2m} + 4r_2^m r_1^m \cos m\phi_2 \cos m\phi_1) \\ & + 2(x_3^m + x_2^m + x_1^m) r_2^m r_1^m (r_2^m \cos m\phi_1 + r_1^m \cos m\phi_2) + r_2^{2m} r_1^{2m}, \end{aligned}$$

$$\begin{aligned} a'_5 = {} & x_3^m x_2^m x_1^m (r_2^{2m} + r_1^{2m} + 4r_2^m r_1^m \cos m\phi_2 \cos m\phi_1) \\ & + 2(x_3^m x_2^m + x_3^m x_1^m + x_2^m x_1^m) r_2^m r_1^m (r_2^m \cos m\phi_1 + r_1^m \cos m\phi_2) \\ & + (x_3^m + x_2^m + x_1^m) r_2^{2m} r_1^{2m}, \end{aligned}$$

$$\begin{aligned} a'_6 = {} & 2x_3^m x_2^m x_1^m r_2^m r_1^m (r_2^m \cos m\phi_1 + r_1^m \cos m\phi_2) \\ & + (x_3^m x_2^m + x_3^m x_1^m + x_2^m x_1^m) r_2^{2m} r_1^{2m}, \end{aligned}$$

$$a'_7 = (x_3 x_2 x_1 r_2^2 r_1^2)^m.$$

Suppose first that

$$|x_3| > |x_2| > |x_1| > r_2 > r_1.$$

Then the dominant equation of (197) is

$$\begin{aligned} & x^7 + x_3^m x^6 + x_3^m x_2^m x^5 + x_3^m x_2^m x_1^m x^4 \\ & \quad + 2x_3^m x_2^m x_1^m r_2^m \cos m\phi_2 x^3 + x_3^m x_2^m x_1^m r_2^{2m} x^2 \\ & \quad + 2x_3^m x_2^m x_1^m r_2^{2m} r_1^m \cos m\phi_1 x + (x_3 x_2 x_1 r_2^2 r_1^2)^m = 0. \end{aligned} \tag{198}$$

Suppose secondly that

$$r_2 > r_1 > |x_3| > |x_2| > |x_1|.$$

Then the dominant equation of (197) is

$$\begin{aligned} & x^7 + 2r_2^m \cos m\phi_2\, x^6 + r_2^{2m} x^5 + 2r_2^{2m} r_1^m \cos m\phi_1 x^4 + r_2^{2m} r_1^{2m} x^3 \\ & \quad + r_2^{2m} r_1^{2m} x_3^m x^2 + r_2^{2m} r_1^{2m} x_3^m x_2^m x + (r_2^2 r_1^2 x_3 x_2 x_1)^m = 0. \end{aligned} \tag{199}$$

Suppose finally that

$$r_2 > |x_3| > r_1 > |x_2| > |x_1|.$$

Then the dominant equation of (197) is

$$x^7 + 2r_2^m \cos m\phi_2 x^6 + r_2^{2m}x^5 + r_2^{2m}x_3^m x^4 + 2r_2^{2m}x_3^m r_1^m \cos m\phi_1 x^3$$
$$+ r_2^{2m}x_3^m r_1^{2m}x^2 + r_2^{2m}x_3^m x_2^m r_1^{2m}x + x_3^m x_2^m x_1^m r_2^{2m} r_1^{2m} = 0. \quad (200)$$

Inspection of Eqs. (198–200), and similar equations, leads to the following conclusions:

As the root-squaring process is continued, the number of coefficients which eventually fluctuate in sign is equal to the number of pairs of complex roots. The $2m$th power of the modulus of any complex root is the quotient of the coefficients on either side of the irregular coefficient. The mth power of one of the real roots is the quotient of two adjacent regular coefficients. Only the regular coefficients are used in the calculation of the moduli, and the root-squaring process is stopped by the application of the rule following Eq. (174) to the regular coefficients.

The real and imaginary parts of the complex roots are determined from the moduli and the coefficients of the original equation in the following manner. Let the complex Encke roots of (196) be written

$$r_k e^{\pm i\phi_k} = u_k \pm iv_k, \qquad (k = 1,2).$$

Multiplying out the products

$$(x + x_3)(x + x_2)(x + x_1)(x + u_1 + iv_1)(x + u_1 - iv_1)$$
$$(x + u_2 + iv_2)(x + u_2 - iv_2) = 0 \quad (201)$$

and equating the coefficients of x^6 and x in (201) and (196) we have the relations:

$$(x_3 + x_2 + x_1) + 2(u_2 + u_1) = a_1,$$
$$2x_3x_2x_1(u_2r_1^2 + u_1r_2^2) + r_2^2r_1^2(x_3x_2 + x_3x_1 + x_2x_1) = a_6. \quad (202)$$

Since u_2 and u_1 are the only unknowns in (202), these equations are sufficient to determine u_2 and u_1, except possibly for sign. The values of v_2 and v_1 are determined by

$$v_k = \pm\sqrt{r_k^2 - u_k^2} \qquad (k = 1,2).$$

47. Rules for Graeffe's Method. The three steps in the application of Graeffe's general theory to all algebraic equations are: (*a*) *Construction of the table;* (*b*) *termination of the process of root-squaring;* (*c*) *calculation of the roots from the table.* **Steps (a) and (b) are the**

same for all equations regardless of the nature of the roots, but step (c) depends upon the nature of the roots.

(*a*) *Construction of the table.* Construct the numerical table by means of the law immediately following Eq. (169). As the successive equations are formed, any coefficient will behave in one of four ways. If m is sufficiently large, a coefficient in the next derived equation of the table may:

(1) Be practically the square of its value in the preceding equation. The coefficient is then said to be regular and the value approached is called its expected value. A coefficient is called irregular if it is not regular.

(2) Approach $1/q$ times its expected value. The degree of the equation is n and $q = 2$ or 3 or 4 or . . . n.

(3) Fluctuate in algebraic sign and at the same time not approach its expected numerical value.

(4) Fail to follow any obvious law of variation.

(*b*) *Termination of the process.* Cease doubling m in the table when the regular coefficients are the squares of those of the preceding equation to the accuracy required and when the coefficients described in a(2) above, if any such are present, are

$$\frac{1}{q}\text{th } (q = 2 \text{ or } 3 \text{ or } 4 \ldots \text{ or } n)$$

of the expected values to the accuracy required.

(*c*) *Calculation of the roots.* Only the absolute value of a root is obtained from the table. The algebraic sign of a root is determined by substitution in the original equation.

Case 1. All roots real and distinct. In this case, all coefficients of the table are regular. The roots are obtained from Eqs. (171), (173), etc., or, what is the same thing, the quotient of each power of x divided by the coefficient of the next higher power of x gives the mth power of a root.

Case 2. All roots real, two or more coincident or equal in absolute value. If the coefficient of x^{n-s} in successive equations is only $1/2\left[\text{or } \frac{1}{q}\text{th}\right]$, the square of the coefficient of x^{n-s} in the preceding equation, then the s and $(s + 1)$ [or the s, $s + 1$, $s + 2$, . . . $s + (q - 1)$] roots are equal in absolute value. The $2m$th power of the multiple root of multiplicity q is $\dfrac{2!(q-2)!}{q!}$ times the quotient of the coefficients

on either side of the irregular coefficient, i.e., the coefficient of x^{n-s}. (It is understood in this section that the coefficient of the lower power of x is in the numerator.) It is supposed now that the 1st, 2nd, . . . s, $s + 1, \ldots s + (q - 1)$ roots have been found. Suppose the $(s + q)$th root is smaller than the multiple root. The mth power of the (s + q)th root is equal to the quotient of the coefficients of $x^{n-(s+q)}$ and $x^{n-(s+q-1)}$. The remaining roots $s + q + 1$, $s + q + 2, \ldots n$ are, found as in case 1. (See Ex. 7, § 47.)

Case 3. One or more real roots and one pair of complex roots. The fluctuation in sign, for m sufficiently large, of one coefficient of the table indicates the presence of one pair of complex roots. The $2m$th power of the modulus of these roots is equal to the quotient of the coefficients on either side of the irregular term. The real and imaginary parts of the complex root are determined by the equations following (180). The real roots are found from the remaining coefficients as in case 2.

Cases 4 and 5. Four or six distinct complex roots. If all the roots are complex and distinct, the coefficients of odd powers of x fluctuate in sign. All other coefficients are regular. The modulus of any one of the roots is equal to the quotient of the coefficients on either side of the irregular term. The real parts of the roots are determined by means of Eqs. (183–185) or (189–191) and the imaginary parts by the relation

$$v_k = \pm\sqrt{r_k^2 - u_k^2} \quad (k = 1, 2 \text{ or } 3).$$

If all the roots of an nth-degree equation are complex and distinct, the procedure is the same. However, the relations between the real parts of the roots and the coefficients of the original equation must be determined. This can be done as indicated in Eq. (182).

Case 6. One real and two coincident complex roots. An irregular coefficient, which apparently follows no law of variation and which stands between two coefficients fluctuating in sign, indicates one pair of coincident complex roots. The $4m$th power of the modulus of the complex root is equal to the quotient of the coefficients on either side of the three irregular coefficients. The constant term of the equation is equal to $r^{4m}x_1^{m}$. The real and imaginary parts of the roots are determined by

$$4u - x_1 = \text{coefficient of } x^4,$$

Case 7. General case. The number of coefficients which fluctuate in sign is equal to the number of pairs of distinct complex roots. The

$2m$th power of the modulus of any distinct complex root is the quotient of the coefficients on either side of the irregular coefficient.

The mth power of one of the distinct real roots is the quotient of two adjacent regular coefficients.

The $2m$th power of a real root of multiplicity q is equal to $\frac{2!(q-2)!}{q!}$ times the quotient of the coefficients on either side of the irregular coefficient satisfying the description in a(2). (Also see case 2.)

Complex double roots are indicated by the presence of three adjacent irregular coefficients, the outside two of which fluctuate in sign. The $4m$th power of the modulus of the double roots is equal to the quotient of the coefficients on either side of the group of three irregular coefficients.

The real parts of the complex roots are obtained through the relations between roots and coefficients of the original equation. (See Eqs. (201–202).)

The imaginary parts of the complex roots are then found as in case 6.

Any prescribed accuracy may be attained by Graeffe's method. If great accuracy is required, it may be advantageous to make a preliminary determination of the roots in order to see how many figures must be retained in squaring and multiplication processes to secure the accuracy required. If the equation is of high degree, it is advantageous to depress the original equation by removing all the real roots. High accuracy for the real roots may be obtained, with little labor, by the methods of §§ 42–43. References by which the accuracy of complex roots can be improved are found at the end of the text. (See Ref. 25.)

EXERCISES

1. Solve the equations:

(a) $x^3 - 7x + 7 = 0.$
(b) $x^3 - 3.5x + 1.6 = 0.$
(c) $x^3 + x^2 - 2x - 1 = 0.$
(d) $\cos x(e^x + e^{-x}) + 2 = 0.$
(e) $x^2 + 4 \sin x = 0.$
(f) $\cos x - 3x + 1 = 0.$

2. Solve the equations:

(a) $x^4 + 2x^3 - 12x^2 - 10x + 3 = 0.$
(b) $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0.$
(c) $x^4 - 11.70x^3 + 5.96x^2 + 131.82x - 214.20 = 0.$
(d) $x^4 + x^3 - 5x^2 + 9x - 10 = 0.$

3. Solve the equation:

$$x^5 + x^4 - 33x^3 - 26x^2 + 256x + 224 = 0.$$

4. Solve the equation:

$$x^7 - 6x^5 + 21x^4 - 21x^3 + 32x^2 - 56x + 18 = 0.$$

5. In the calculation of the transient field current of a synchronous machine by Heaviside's expansion theorem, it was necessary to find the roots of

$$p^5 + 0.6193p^4 + 0.1273p^3 + 1.044 \times 10^{-2}p^2 + 2.966 \times 10^{-4}p + 5.739 \times 10^{-7} = 0.$$

What are these roots?

6. The roots of the characteristic equation of a simultaneous system of linear differential equations with constant coefficients characterize the motion (or currents) of the physical system. If the real roots and real parts of the complex roots are negative or zero, the motion is limited in magnitude and is said to be **stable.** On the other hand, if the roots have positive real parts, the motion is **unstable.** Frequently, in design, it is sufficient to know whether a system is stable. The criterion for stability is much more easily applied than Graeffe's method.

The characteristic equation of the differential equations defining the motion of a certain type of electric locomotive was found to be

$$x^6 + 68.6x^5 + 785x^4 + 7213x^3 + 50{,}700x^2 + 8200x + 435{,}000 = 0.$$

(See Vol. II, Chap. I.) It is desired to determine the stability of the motion.

The proof of the following criterion is found in Routh's *Advanced Rigid Mechanics*, p. 170. Criterion: Arrange the coefficients of

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = 0$$

in two rows

$$p_0 \quad p_2 \quad p_4 \dots$$

$$p_1 \quad p_3 \quad p_5 \dots$$

Form a new row by cross-multiplication as follows:

$$\frac{p_1p_2 - p_0p_3}{p_1}, \frac{p_1p_4 - p_0p_5}{p_1}, \dots.$$

Form a fourth row by similar cross-multiplication on the second and third rows. The number of terms in successive rows decreases. Stop when one term is left.

A necessary and sufficient condition that the equation have no roots whose real parts are positive in that all the terms in the first column have the same sign. The number of variations of sign in the first column is equal to the number of roots or pairs of complex roots with their real parts positive.

Since only the signs are important, any row may be divided by a positive constant to simplify multiplication.

Applying the criterion to the locomotive equation we have:

1	785	50,700	435,000
68.6	7,213	8,200	
$\dfrac{68.6 \times 785 - 7{,}213}{68.6}$;	$\dfrac{68.6 \times 50{,}700 - 8{,}200}{68.6}$;	..........;	435,000
6.86	721.3	820	
6.7986	505.8	4,350	
$\dfrac{6.7986 \times 721.3 - 6.86 \times 505.8}{6.7986}$;	$\dfrac{6.7986 \times 820 - 4{,}350 \times 6.86}{6.7986}$;	..........;	
6.7986	505.8	4,350	
2.1092	−35.692		
2.1092	−35.692		
6.2085	43.5		
6.2085	43.5		
−50.469			
−2195.4			

The change of sign occurring in column one indicates the presence of one pair of roots having positive real parts. The motion is unstable, but can be corrected by varying one of the parameters of the coefficients in the characteristic equation. The roots are found to be (approximately)

$$x = -56.7, \quad -11.02, \quad -0.92 \pm j9.07, \quad +0.47 \pm j2.86.$$

Is the motion stable when the characteristic equation is

$$x^4 + 2.55 \times 10x^3 + 6.85 \times 10^2x^2 + 8.15 \times 10^3x + 4.74 \times 10^4 = 0?$$

7. Prove the rule in case 2, § 47.

V

DIMENSIONAL ANALYSIS

The first four sections of the present chapter are seemingly unrelated from a mathematical standpoint, but from an engineering point of view they are closely related since they form a single engineering problem-solving unit. To begin with, the mathematical formulation of the engineering problem may lead to one or more ordinary linear differential equations. Determinants afford systematic methods of manipulating simultaneous systems of differential and algebraic equations. Complicated voltages and forces are expressible in Fourier

series, and these series can be handled successfully in ordinary differential equations. Finally, Graeffe's method gives a quick and accurate method of obtaining the roots of the characteristic equation. These roots then display the general nature of the motions of the system.

A second problem-solving unit, which is independent of the first, is dimensional analysis. In certain respects, dimensional analysis is applicable to a wider range of problems than the methods so far explained. The principles and relations discussed up to this point are defined by systems of ordinary linear differential equations with constant coefficients. The problems solvable by dimensional analysis are subject to no such narrow restrictions. Much information regarding results of problems reducible to ordinary and partial linear and non-linear differential equations can be obtained by dimensional methods. In fact, these methods are applicable also to problems whose solutions do not depend upon differential equations. Moreover, the amount of labor and mathematical knowledge necessary for carrying out a dimensional analysis solution is usually small in comparison with that required for solutions otherwise obtained.

The method, however, has the great disadvantage that its results, in general, give information much less complete than that obtained by the solution of differential equations. However, dimensional results are adequate in many engineering problems.

48. Uses and Nature of Dismensional Analysis. The principal value of dimensional analysis lies in the following applications:

(*a*) Checking equations.
(*b*) Changing units.
(*c*) Derivation of formulas.
(*d*) Analysis of physical systems by use of models or physically similar systems.
(*e*) Systematic experimentation.

These topics are discussed in §§ 50–56.

Owing to the fact that a discussion of dimensional analysis in its complete generality takes on rather abstract aspects, the nature of such analysis is best perceived by the examination of results obtained by it in representative problems. But before writing down such results we first recall from elementary physics some ideas regarding dimensions.

The dimensions of a physical quantity are concerned with its *quality* or *kind* and not with its magnitude. Underlying all dimensional analysis is the concept of the *dimensional formula of a physical quantity*, which is an expression showing the manner in which a group

of chosen fundamental units are combined to make a unit of that quantity. For example, in terms of length as a fundamental unit, the dimensions of area and volume are respectively (length)^2 and (length)^3, or in symbols $[A] = L^2$ and $[V] = L^3$. (This is read: "The dimensions of volume are length cubed.") If in any given problem involving n physical quantities (dimensions), r of them are expressed in terms of the $(n - r)$ remaining ones, the $(n - r)$ quantities are called the **fundamental dimensions** or **primary quantities,** and the r quantities are called **derived dimensions** or **secondary quantities.** For example, in the problem of § 6 the n quantities are acceleration, force, mass, length, velocity, and time. The fundamental quantities may be chosen to be length, mass, and time. Acceleration, force, and velocity are then derived quantities. Not every set of $(n - r)$ dimensions can be used as a fundamental system of units. The criterion for a fundamental system appears later, in § 51. Length L, mass M, and time T, however, form a suitable system for mechanics. The dimensional formula for a quantity is derived from the definition of the quantity.

In mechanics, we may have the following dimensional formulas:

$[\text{Linear displacement } x] = L,$

$\left[\text{Linear velocity } \frac{dx}{dt}\right] = \frac{L}{T} = LT^{-1},$

$\left[\text{Linear acceleration } \frac{d^2x}{dt^2}\right] = LT^{-2},$

$[\text{Linear momentum}] = MLT^{-1},$

$[\text{Angle}] = \frac{L}{L} \text{ (dimensionless)},$

$[\text{Angular velocity}] = T^{-1},$

$[\text{Angular acceleration}] = T^{-2},$

$[\text{Moment of inertia}] = ML^2$

$[\text{Angular momentum}] = ML^2T^{-1},$

$[\text{Kinetic energy}] = ML^2T^{-2},$

$[\text{Force}] = MLT^{-2},$

$[\text{Work}] = ML^2T^{-2},$

$[\text{Power}] = ML^2T^{-3},$

$[\text{Moment of force}] = ML^2T^{-2},$

$[\text{Density } (\rho)] = ML^{-3}.$

In fluid mechanics, we may have the dimensional formulas:

$$\left[\text{Pressure gradient } (G) = \frac{\text{Force}}{\text{Area}} \div \text{Length}\right] = ML^{-2}T^{-2},$$

$$\left[\text{Viscosity } (\mu) = \frac{\text{Force}}{\text{Area}} \div \text{Velocity gradient}\right] = ML^{-1}T^{-1},$$

$$\left[\text{Kinematic viscosity } (\nu) = \frac{\mu}{\text{Density}}\right] = L^2T^{-1}.$$

Suitable primary quantities in heat-flow problems are L, T, H (quantity of heat), and θ (temperature), or L, M, T, and θ. Dimensional formulas for certain secondary quantities in terms of the former are:

$$[\text{Rate of heat transfer}] = HT^{-1},$$

$$[\text{Thermal capacity (per unit volume)}] = HL^{-3}\theta^{-1},$$

$$\left[\text{Thermal conductivity } \frac{\text{(Rate of heat transfer per unit area)}}{\text{Temperature gradient}}\right] = HL^{-1}T^{-1}\theta^{-1},$$

or, in terms of the latter primary quantities,

$$\begin{aligned}
[\text{Rate of heat transfer}] &= ML^2T^{-3},\\
[\text{Thermal capacity}] &= ML^{-1}T^{-2}\theta^{-1},\\
[\text{Thermal conductivity}] &= MLT^{-3}\theta^{-1}.
\end{aligned}$$

Suitable primary quantities in electrical problems are M, L, T and $\mathcal{E}$ (the dielectric constant). Dimensional formulas for certain secondary quantities are:

$$\begin{aligned}
[\text{Charge}] &= \mathcal{E}^{1/2}L^{3/2}M^{1/2}T^{-1},\\
[\text{Voltage}] &= \mathcal{E}^{-1/2}L^{1/2}M^{1/2}T^{-1},\\
[\text{Current}] &= \mathcal{E}^{1/2}L^{3/2}M^{1/2}T^{-2},\\
[\text{Capacitance}] &= \mathcal{E}L,\\
[\text{Resistance}] &= \mathcal{E}^{-1}L^{-1}T.
\end{aligned}$$

The nature of dimensional analysis can now be seen by inspection of the results in representative problems.

49. Some Representative Results. The methods of obtaining the results displayed in this section is the subject-matter of subsequent sections.

1. *Checking equations.* The differential equations of problem 1, § 20, are (where primes indicate derivatives with respect to time):

$$M_2s_2'' + k_d(s'_2 - s'_1) + k_2(s_2 - s_1) = 0,$$

$$M_1s_1'' - k_d(s'_2 - s'_1) + k_1s_1 - k_2(s_2 - s_1) = 0,$$

where M_1 and M_2 are given in slugs, k_d in pounds per unit velocity, and k_1 and k_2 in pounds per unit displacement. Let it be required to check dimensionally these equations. Owing to similarity of various terms, it is necessary to check only three. Since

$$[M_2s''_2] = MLT^{-2}$$

$$]k_d s'_2] = \frac{MLT^{-2}}{LT^{-1}} \times \frac{L}{T} = MLT^{-2}$$

$$[k_2 s_2] = \frac{MLT^{-2}}{L} \times L = MLT^{-2}$$

the terms have the same dimensions and the equations are dimensionally correct.

2. *Changing units.* Let it be required to convert Q kilowatts to it equivalent in feet, pounds, and seconds. By simple consideration of dimensions, the result can be shown to be

$$Q \text{ kilowatts} = Q7370 \frac{\text{foot-pounds}}{\text{seconds}}.$$

3. *Derivation of formula: Oscillation of rotor.* Eq. (140), § 32, may be written

$$I\theta'' + T_d(\theta' - \omega) + T_s(\theta - \omega t) = f(t).$$

Suppose there are no torques on the rotor except the synchronizing and inertial torques, and let the synchronizing torque, $T_s(\theta - \omega t)$, be replaced by its more accurate value $T_s \sin (\theta - \omega t)$. Replacing $(\theta - \omega t)$ by a new variable ϕ, the equation becomes

$$I\phi'' + T_s \sin \phi = 0.$$

If the rotor is displaced from synchronous position and suddenly released, find its period of oscillation. The period t can be found by dimensional analysis to be

$$t = \sqrt{\frac{I}{T_s}}\, F(\phi_0),$$

where ϕ_0 is the amplitude of the periodic variation of ϕ. The quantity ϕ_0, since it is an angle, is dimensionless. The function $F(\phi_0)$, which is a function of a dimensionless argument, cannot be found by dimensional methods. Its behavior is easily determined experimentally, since for I and T_s fixed, t is a function of ϕ_0 alone.

4. *Derivation of formula: Amplitude of damped oscillation of a mass.* Let the mass in Fig. 2, § 10, be acted upon by the vertical force $F \sin \omega t$. Let the damping and spring constants be respectively k_d and k. By dimensional analysis, it is easily shown that the amplitude of vibration A is

$$A = \frac{F}{k} f\left(\frac{k}{M\omega^2}, \frac{\omega k_d}{k}\right).$$

The arguments, $\frac{k}{M\omega^2}$ and $\frac{\omega k_d}{k}$, are dimensionless, and the function f cannot be determined by dimensional reasoning. The nature of f can be determined experimentally.

In this case, the function f is easily found by other means (from the solution of the differential equation) to be

$$f\left(\frac{k}{M\omega^2}, \frac{\omega k_d}{k}\right) = \frac{1}{\left[\left(1 - \frac{M\omega^2}{k}\right)^2 + \left(\frac{\omega k_d}{k}\right)^2\right]^{1/2}}.$$

In this example, much more complete information is easily obtained by the solution of the differential equation than by dimensional analysis.

5. *Derivation of formula: Propeller thrust.* It is desired to determine the thrust T of a screw propeller, which is so deeply immersed in water that there is no surface turbulence. The result for this problem is

$$T = \rho D^2 S^2 f\left(\frac{Dn}{S}, \frac{DS}{\nu}, \frac{Dg}{S^2}\right),$$

where: D = diameter of the propeller,
n = revolutions per minute,
S = speed of advance,
g = acceleration of gravity,
ρ = mass density of water,
ν = kinematic viscosity of water.

The function f can be determined experimentally.

6. *Derivation of formula: Heat transfer.* A solid body of certain shape is held at a constant temperature θ and fixed in a stream of liquid which flows past it at a velocity V. It is required to find the rate of heat transfer R from the body to the liquid. If viscosity and surface conditions can be neglected, the result in this problem is

$$R = k(\Delta\theta)lf\left[\left(\frac{lVc}{k}\right), r, r', r'', \ldots\right],$$

where k = thermal conductivity of the liquid,
$\Delta\theta$ = temperature difference,
l = a linear dimension (say the length),
V = velocity of the stream,
c = heat capacity of liquid per unit volume,
$r, r', r'' \ldots$ = ratios of dimensions of body (shape factors), and f is an unknown function.

The quantities l, c, k, and r, r', r'' . . . are constant for a given body and liquid. Thus the function f might be determined by measuring $\frac{R}{\Delta\theta}$ as V varies.

7. *Derivation of formula: Flow in smooth pipes.* Liquid flows, at a constant rate, through a smooth, straight pipe. Let it be required to find the pressure gradient G as a function of the diameter D of the pipe, the speed S, density ρ, and viscosity μ of the liquid. The result turns out to be

$$G = \frac{\rho S^2}{D} f\left(\frac{DS\rho}{\mu}\right).$$

8. *Derivation of formula: Velocity of sound in a gas.* It is easily shown by dimensional analysis that the velocity V of sound in a gas is

$$V = \text{constant} \sqrt{\frac{p}{\rho}},$$

where p and ρ are respectively the pressure and density of the gas.

9. *Derivation of formula: Air resistance on airplane wing.* It is known from observations that the resistance R, which the air offers to the wing of an airplane, depends primarily on the shape, size L, and speed V of the wing and on the density ρ and viscosity μ of air. The formula for the resistance R is, by dimensional methods,

$$R = \rho V^2 L^2 f\left(\frac{\rho VL}{\mu}\right).$$

10. *Derivation of formula: Period of oscillatory circuit.* An oscillatory discharge is excited in a simple series circuit possessing capacitance C and inductance L. By the methods of this section the period t of oscillation is found to be

$$t = \text{constant} \sqrt{LC}.$$

11. *Derivation of formula: Energy density of electromagnetic field.* Suppose that the energy density u is completely determined by the field strengths H and E, and by the permeability μ and dielectric constant ε of the isotropic medium. If E, ε, μ are taken as primary quantities, it is found that

$$u = \varepsilon E^2 f_1\left(\frac{\mu H^2}{\varepsilon E^2}\right).$$

But if H, $\mathcal{E}$, μ are taken as primary quantities, the equivalent result

$$u = \mu H^2 f_2\left(\frac{\mathcal{E}E^2}{\mu H^2}\right)$$

is obtained.

12. *Use of models: Water resistance to moving ship.* Results 3–11 inclusive are examples of formulas derivable by dimensional analysis. As a final result, we indicate how the behavior of a physical system may be predicted by a study of a model. For example, in naval architecture, the water resistance to the motion of full-sized ships is predicted by measuring the resistance on models. By the methods here considered it can be shown that the force of skin resistance P, which a fluid (water) offers to a ship in motion, is

$$P = \rho S V^2 f\left(\frac{\nu}{lV}\right),$$

where the symbols have the significance:

ρ = density of the fluid,
S = immersed surface,
l = a linear dimension (say the width),
V = velocity of the ship,
ν = kinematic viscosity of water,
P = force on the ship.

If the same symbols with primes denote similar quantities for a model which has the same shape as the ship, then

$$P' \quad \rho' S' V'^2 f\left(\frac{\nu'}{l'V'}\right),$$

where the unknown functional relationship f is *the same* for both ship and model. If the model is run in water so that $\rho' = \rho$ and $\nu' = \nu$ then the ratio P/P' is

$$\frac{P}{P'} = \frac{S V^2 f\left(\frac{\nu}{lV}\right)}{S' V'^2 f\left(\frac{\nu'}{l'V'}\right)}.$$

Now the function f is unknown but is the same in the numerator and denominator. We can eliminate it from the ratio if V' is taken such that

$$\frac{\nu}{lV} = \frac{\nu}{l'V'} \quad \text{or} \quad V' = \frac{l}{l'} V.$$

Thus, if $V' = \frac{lV}{l'}$, the resistance on the full-sized ship is

$$P = \frac{P'S}{S'V'^2} V^2,$$

where the numerical values of the primed quantities are obtained from experiments on the model.

Inspection of the results of this section leads naturally to the question, What is the general theory by which they are obtained? Let us consider the theory in the order of its applications listed in § 48.

50. Checking Equations. Fourier stated the principle that all terms of a physical equation must have the same dimensions. Let $Q_1, Q_2, \ldots Q_n$ be n physical quantities (say length, viscosity, etc.), which are involved in some physical phenomenon. An equation, in $Q_1, Q_2, \ldots Q_n$, describing a relation or motion of a physical system is called a **physical equation.**

That the principle stated by Fourier is not necessarily true may be seen as follows. Consider the relations for a body falling from rest under the influence of gravity which involve distance fallen s, velocity v, and time t. We have $s = \frac{1}{2}gt^2$ and $v = gt$, and by adding these two equations, we obtain

$$s + v = gt + \tfrac{1}{2}gt^2,$$

not all the terms of which have the same dimensions. This is a physical equation (according to the definition above), and obviously, by this definition, the principle stated by Fourier is false.

This leads to the introduction of additional definitions and to a restatement of the principle. The last equation is true no matter how the magnitudes of the primary units involved are changed. An equation which has this property is called a **complete** physical equation. If a physical equation is complete and if there exist no other relations between the quantities in the problem considered except those given by the equation, then all terms of the equation have the same dimensions, and the equation is said to be **dimensionally homogeneous.**

The equation of the above example is complete, but since there are certain other relations between the quantities involved, it is not dimensionally homogeneous. There exist true equations which are not even complete. Such equations are called **adequate** equations. Examples are given later. (See § 52.) In engineering, nearly all equations are dimensionally homogeneous, and we thus expect all

terms to have the same dimensions or we expect to determine the reason why they do not.

We thus make use of Fourier's statement (with the proper reservations) in eliminating errors and in remembering equations and formulas. Eq. (140) can be written

$$I\theta'' + T_d\theta' + T_s\theta = T_d\omega + T_s\omega t + f(t),$$

where:
θ = angular displacement,
I = moment of inertia,
T_d = torque per difference of angular velocities,
T_s = torque per difference of angular positions,
$f(t)$ = applied torque.

In view of the dimensional formulas of § 48, the terms of the equation have the following dimensions:

$$[I\theta''] = ML^2 \times T^{-2} = ML^2T^{-2},$$

$$[T_d\theta'] = \frac{MLT^{-2} \times L}{T^{-1}} \times T^{-1} = ML^2T^{-2},$$

$$[T_s\theta] = MLT^{-2} \times L \times \frac{L}{L} = ML^2T^{-2},$$

$$[T_d\omega] = \frac{MLT^{-2} \times L}{T^{-1}} \times T^{-1} = ML^2T^{-2},$$

$$[T_s\omega t] = MLT^{-2} \times L \times T^{-1} \times T^1 = ML^2T^{-2},$$

$$[f(t)] = MLT^{-2} \times L = ML^2T^{-2}.$$

Thus the equation is dimensionally correct.

51. Change of Units. Change of units in all cases is reducible to a routine process, that is, to either the application of a simple rule or formula. There are two types of change of unit. In the first, merely the magnitudes of the measuring units are changed, for example, in the reduction of feet per second to miles per hour. Changes of unit of the first type are always possible. In the second type, not only are the magnitudes of the measuring units changed, but also the character or kinds of units are changed. For example, the primary quantities may be changed from length, time, and mass to length, time, and force. Change of units of the second type are possible only under certain conditions. Obviously a linear velocity, which is expressible in miles per hour, is not expressible also in revolutions per second. Thus a necessary and sufficient condition for a change of units of the second type is desired.

(*a*) *Change of units of the first type.* The following rule for change of units of the first type is obvious.

RULE: **Write the dimensional formula for the physical quantity considered, treating the dimensional symbols (or groups of them) as the name of concrete things. Replace each symbol (or group) by its equivalent in terms of the new unit of measure which is to replace it.**

EXAMPLE. Reduce Q foot-pounds per hour to ergs per second.

$$\frac{Q \text{ foot-pounds}}{\text{hour}} = \frac{(\text{feet})(\text{pounds})}{(\text{hours})} Q$$

$$= \frac{(30.48 \text{ cm.})(444{,}820 \text{ dynes})}{(3600 \text{ sec.})} Q$$

$$= 3766.1\, Q \frac{\text{centimeter-dynes}}{\text{seconds}} = \frac{3766.1\, Q \text{ ergs}}{\text{seconds}}.$$

(*b*) *Change of units of the second type.* Let it be required to change from a system of units in which the primary quantities are $X_1, X_2, \ldots X_n$ to a system whose primary quantities are $Y_1, Y_2, \ldots Y_n$. In the proof for simplicity of notation, n is taken to be 3. Denote the physical quantity considered by Q. Suppose the dimensional formula[13] for Q is

$$[Q] = X_1^{a_1} X_2^{a_2} X_3^{a_3}.$$

Then

$$Q = h_0 X_1^{a_1} X_2^{a_2} X_3^{a_3}, \tag{206}$$

where h_0, a_1, a_2, and a_3 are numerical constants. Let the equations which relate the new primary quantities to the old be

$$\left.\begin{aligned} H_1 Y_1 &= X_1^{a_{11}} X_2^{a_{12}} X_3^{a_{13}}, \\ H_2 Y_2 &= X_1^{a_{21}} X_2^{a_{22}} X_3^{a_{23}}, \\ H_3 Y_3 &= X_1^{a_{31}} X_2^{a_{32}} X_3^{a_{33}}, \end{aligned}\right\} \tag{207}$$

where H_i and a_{ij} $(i, j = 1, 2, 3)$ are constants. Eqs. (207) may be written,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= h_1 y_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= h_2 y_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= h_3 y_3, \end{aligned}\right\} \tag{208}$$

[13] In § 48 all dimensional formulas were seen to be products. This is true for all quantities of physics.

where
$$x_i = \log X_i,$$
$$h_i y_i = \log H_i Y_i \ (i = 1, 2, 3).$$

The solution for x_1 of (208) by Cramer's rule (§26) is

$$x_1 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \frac{h_1 y_1}{\Delta} - \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \frac{h_2 y_2}{\Delta} + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \frac{h_3 y_3}{\Delta},$$

where Δ is the determinant of the system. The value of X_1 then is

$$X_1 = (H_1 Y_1)^{+\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{\frac{1}{\Delta}}} \cdot (H_2 Y_2)^{-\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}^{\frac{1}{\Delta}}} \cdot (H_3 Y_3)^{+\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}^{\frac{1}{\Delta}}}.$$

Similarly,

$$X_2 = (H_1 Y_1)^{-\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}^{\frac{1}{\Delta}}} \cdot (H_2 Y_2)^{+\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}^{\frac{1}{\Delta}}} \cdot (H_3 Y_3)^{-\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}^{\frac{1}{\Delta}}},$$

$$X_3 = (H_1 Y_1)^{+\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}^{\frac{1}{\Delta}}} \cdot (H_2 Y_2)^{-\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}^{\frac{1}{\Delta}}} \cdot (H_3 Y_3)^{+\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^{\frac{1}{\Delta}}}. \tag{209}$$

When these values of X_1, X_2, and X_3 are substituted in (206) the physical quantity Q is expressed in terms of the new units Y_1, Y_2, and Y_3.

Consider a change from a set of fundamental units consisting of mass in pounds (mass), length in miles, and time in hours to another set consisting of power in kilowatts, velocity in feet per second, and energy in foot-pounds. The first three units correspond to X_1, X_2, and X_3 of Eq. (206), and the last three to Y_1, Y_2, and Y_3. We desire to find the magnitude of any physical quantity such as momentum in terms of the second set of units if it has been given in terms of the first set.

Eqs. (207), which relate the new units to the old, become

$$Y_1 = (1 \text{ kw.}) = \frac{33{,}000}{0.746} \frac{\text{ft-lb.}}{\text{min.}}$$

$$= \frac{33{,}000}{0.746} \left[\frac{32.17 \text{ lb.} \left(\frac{\text{mi.}}{5280}\right)^2}{\frac{\text{hr.}}{60} \left(\frac{\text{hr.}}{3600}\right)^2} \right]$$

$$= 3.97 \times 10^7 \text{ lb.}^1 \text{ mi.}^2 \text{ hr.}^{-3}$$

$$= 3.97 \times 10^7 X_1 X_2^2 X_3^{-3},$$

$$Y_2 = 1 \text{ ft. per sec.} \quad \frac{3600}{5280} \text{ lb.}^0 \text{ mi.}^1 \text{ hr.}^{-1},$$

$$Y_3 = 1 \text{ ft-lb.} = \text{ft.} \left[\frac{32.17 \text{ lb. (mass)} \times \text{ft.}}{\text{sec.}^2}\right]$$

$$32.17 \frac{\text{lb.} \left(\dfrac{\text{mi.}}{5280}\right)^2}{\left(\dfrac{\text{hr.}}{3600}\right)^2}$$

$$14.96 \text{ lb.}^1 \text{ mi.}^2 \text{ hr.}^{-2},$$

or

$$\text{lb.}^1 \text{ mi.}^2 \text{ hr.}^{-3} = 2.52 \times 10^{-8} \, Y_1,$$

$$\text{lb.}^0 \text{ mi.}^1 \text{ hr.}^{-1} = 1.47 \, Y_2,$$

$$\text{lb.}^1 \text{ mi.}^2 \text{ hr.}^{-2} = 6.68 \times 10^{-2} \, Y_3.$$

Solving for pounds, miles, and hours by Eqs. (209), we have

$$\text{lb.} = 3.10 \times 10^{-2} \quad Y_2^{-2} Y_3^1 = 3.10 \times 10^{-2} \frac{\text{ft-lb.}}{(\text{ft./sec.})^2},$$

$$\text{mi.} = 3.90 \times 10^6 \quad Y_1^{-1} Y_2 Y_3 = 3.90 \times 10^6 \frac{(\text{ft-lb.})(\text{ft./sec.})}{\text{kw.}}$$

$$\text{hr.} = 2.66 \times 10^6 \quad Y_1^{-1} Y_3^1 = 2.66 \times 10^6 \frac{(\text{ft-lb.})}{\text{kw.}}.$$

N_1 units of momentum in the first set of units are equal in the second set of units to

$$N_1 \frac{\text{lb. (mass) mi.}}{\text{hr.}} = N_1 \, 4.55 \times 10^{-2} \, Y_2^{-1} \, Y_3 = 4.55 \times 10^{-2} N_1 \frac{\text{ft-lb.}}{\text{ft./sec.}}.$$

Likewise the conversion for N_2 units of kinetic energy is

$$N_2 \frac{\text{lb. (mass) mi.}^2}{(\text{hr.})^2} \quad N_2 \, 6.68 \times 10^{-2} \, Y_3 = 6.68 \times 10^{-2} N_2 \text{ ft-lb.}$$

And the conversion for N_3 units of power is

$$N_3 \frac{\text{lb. (mass) mi.}^2}{(\text{hr.})^3} = N_3 \, 2.52 \times 10^{-8} \, Y_1 = 2.52 \times 10^{-8} N_3 \text{ kw.}$$

(*c*) *Criterion for change of units of the second type.* A necessary and sufficient condition for a change of units of the second type to be possible is that Eqs. (208) have a unique solution. By reference to § 26, we see that (208) has a unique solution if and only if $\Delta \neq 0$. In the last example $\Delta = 1$, and the transformation proposed was valid.

Thus we also have a criterion as to whether certain quantities may be taken as primary quantities.

EXAMPLE. Let the transformation required be the reduction of linear miles per hour to radians per second. Eqs. (207) become

$$1 \text{ radian} = (\text{miles})^0 \left(\frac{\text{hours}}{3600}\right)^0,$$

$$1 \text{ second} = (\text{miles})^0 \left(\frac{\text{hours}}{3600}\right)^1.$$

Evidently

$$\Delta = \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} = 0$$

and the transformation is, of course, impossible.

52. Dimensional Constants. A number of dimensional formulas have been listed in § 48, but there are, of course, many more. In addition to dimensional formulas for physical quantities, there are dimensional formulas for certain constants. One of these is the gravitational constant. In the universal law of gravitation,

$$F = \frac{Gm_1m_2}{d^2},$$

G is a dimensional constant. (F is the gravitational force of attraction between two bodies of masses m_1 and m_2 a distance d apart.) Since

$$[F] \quad \frac{ML}{T^2}$$

and

$$\left[\frac{m_1m_2}{d^2}\right] = \frac{M^2}{L^2}$$

it follows that

$$[G] = M^{-1}L^3T^{-2}.$$

A constant which has dimensions and which changes in numerical value when a change of units is made in the equation in which it is found is called a **dimensional** constant.

In the fundamental theorem (the Buckingham π theorem) of dimensional analysis, dimensional constants play the same rôle as secondary quantities.

It is provable that all dimensional formulas and dimensional constants are products of powers of the primary units employed.

It is now possible to give examples of "adequate," "complete," and "dimensionally homogeneous" equations. If in the two-body problem the unit of mass is $m_1 + m_2$ and if the unit of length is the distance between m_1 and m_2, and further if the unit of time is properly chosen, then the universal law of gravitation is

$$\text{(Canonical units of force) } F = \frac{m_1 m_2}{d^2}$$

This equation is not true for arbitrary changes of units in L, M, and T. It is an adequate equation. In § 50, an example of a complete equation has been given. Also the equation

$$F = \frac{G m_1 m_2}{d^2},$$

where

$$[G] = M^{-1}L^3T^{-2}$$

is a complete equation. It is true for all changes of M, L, and T. It happens to be also dimensionally homogeneous.

53. Introductory Problem Leading to the π Theorem. The Buckingham π theorem occupies the same central position in dimensional analysis that Newton's laws of motion occupy in mechanics. We now solve a simple problem which leads up to this theorem.

Let it be required to obtain result 3 of § 49. In this problem it is known from experience that the period of swing t depends primarily upon the moment of inertia I, the synchronizing torque per unit of angular displacement T_s, and the amplitude of swing ϕ_0. *Suppose*, first, that the period of swing can be expressed as the product of powers of these variables, that is

$$t = I^x T_s^y \phi_0^z,$$

where x, y, and z are unknown. This is a physical equation, and we suppose, secondly, that it is dimensionally homogeneous. In other words we assume that both members have the same dimension, M^0L^0T or T. From §§ 48–50

$$[I] = ML^2,$$

$$[T_s] = MLT^{-2} \times L = ML^2T^{-2},$$

ϕ_0 is dimensionless.

By the condition of dimensional homogeneity

$$[t] = [I^x T_s^y \phi_0^z],$$

or

$$M^0L^0T = (ML^2)^x(ML^2T^{-2})^y(M^0L^0T^0)^z.$$

Thus equating the exponents of M, L, T on either side

$$0 = x + y,$$

$$0 = 2x + 2y,$$

$$1 = - 2y.$$

Thus $x = \frac{1}{2}$, $y = - \frac{1}{2}$, and z is indeterminate. Hence

$$= \sqrt{\frac{I}{T_s}}\,\phi_0^z,$$

where z can have any value. Moreover, if z is assigned a number of values, a linear combination of the corresponding terms is also dimensionally homogeneous. That is

$$t = \sqrt{\frac{I}{T_s}} \sum_n A_n \phi_0^n,$$

where the A_n are constants, is also a solution. Since the series can represent any function we may write

$$= \sqrt{\frac{I}{T_s}}\, F(\phi_0),$$

where $F(\phi_0)$ is any function of ϕ_0. This is the result desired. Since $F(\phi)$ is a dimensionless function, dimensional methods can yield no further information. If the last equation is divided by $\sqrt{I/T_s}$ the left side of the resulting equation is a dimensionless product.

$$t\sqrt{\frac{T_s}{I}} = F(\phi_o).$$

In the solution of this problem, two assumptions were made. These assumptions were made because they are justified by the π theorem. $F(\phi_o)$ is determined experimentally.

54. The π Theorem. Let us denote by (α, β, γ . . . to n quantities) a set of measurable quantities and dimensional constants. The total number of primary quantities in terms of which the n quantities (α, β, γ . . .) can be expressed is m. The π theorem is: " If the equa-

tion $f(\alpha, \beta, \gamma \ldots) = 0$ is to be a complete equation, the solution has the form

$$F(\pi_1, \pi_2, \ldots) = 0,$$

where the π's are the $n - m$ independent products of the arguments $\alpha, \beta, \gamma \ldots$ which are dimensionless in the fundamental units." The equation $F(\pi_1, \pi_2, \ldots) = 0$ can be solved for α (say) obtaining

$$\alpha = \beta^{k_1}\gamma^{k_2} \ldots \phi(\pi_2 \pi_3, \ldots).$$

The use of this theorem is to predict a form of the result of a problem.

Let us interpret the π theorem in terms of the introductory example. The complete equation desired is

$$f(t, I, T_s, \phi_0) = 0.$$

The n quantities $(\alpha, \beta, \gamma \ldots)$ are t, I, T_s, and ϕ_0. Two dimensionless products can be formed

$$\pi_1 = t\sqrt{\frac{T_s}{I}},$$

$$\pi_2 = \phi_0.$$

The equation

$$F(\pi_1, \pi_2, \ldots) = 0$$

is in this case

$$t\sqrt{\frac{T_s}{I}} - F(\phi_0) = 0.$$

Finally, the equation

$$\alpha = \beta^{k_1}\gamma^{k_2} \ldots \phi(\pi_2, \pi_3, \ldots)$$

is

$$t = \sqrt{\frac{I}{T_s}}\, F(\phi_0).$$

A proof of the π theorem contributes very little, if any, to its successful application in engineering work. Consequently no proof is given here, but reference to Buckingham's original proof is found at the end of the text, Ref. 28. A number of applications of the π theorem follow, in which certain fine points in the application of the theorem are pointed out. The fine points are thus, perhaps, more easily understood than if couched in a general discussion.

Example 1. Let us find, by dimensional analysis, an expression for the resisting force R which the air offers to the wing of an airplane. It is known from observations that the resistance depends only on (1)

shape, (2) size l, (3) speed v, (4) density of air ρ, and (5) viscosity of air μ. The dimensions of the last four quantities are respectively L, $\frac{L}{T}$, $\frac{M}{L^3}$, and $\frac{M}{LT}$. Thus there are three primary quantities M, L, and T, and five secondary quantities R, v, l, ρ, and μ. By the π theorem, the solution is

$$F(\pi_1, \pi_2) = 0,$$

where π_1 and π_2 are dimensionless products. The general expression for π_1 and π_2 is

$$\pi = \rho^x v^y l^z \mu^w R^u,$$

or

$$[\pi] = \left(\frac{M}{L^3}\right)^x \left(\frac{L}{T}\right)^y (L)^z \left(\frac{M}{LT}\right)^w \left(\frac{ML}{T^2}\right)^u.$$

This expression must be dimensionless. This demands that the equations

$$\begin{aligned} x + w + u &= 0, \\ -3x + y + z - w + u &= 0, \\ -y - w - 2u &= 0, \end{aligned} \tag{210}$$

be satisfied. We have three equations in five unknowns. Two unknowns may be chosen arbitrarily provided the determinant of the remaining three is not zero. (See last paragraph of this example.) For π_1 let $w = 0$, $u = -1$. (These particular values are chosen to make the final solution simpler.) Then the equations become

$$\begin{aligned} x &= 1, \\ -3x + y + z &= 1, \\ -y &= -2, \end{aligned}$$

or

$$x = 1,\ y = 2,\ z = 2,$$

and

$$\pi_1 = \frac{\rho v^2 l^2}{R}.$$

For π_2 choose $u = 0$, $w = -1$. The equations become

$$\begin{aligned} x &= 1, \\ -3x + y + z &= -1, \\ -y &= -1, \end{aligned}$$

from which $x = 1, y = 1, z = 1$, and

$$\pi_2 = \frac{\rho v l}{\mu}.$$

Thus

$$F(\pi_1, \pi_2) = 0,$$

or

$$F\left(\frac{\rho v^2 l^2}{R}, \frac{\rho v l}{\mu}\right) = 0.$$

If we solve for π_1 in terms of π_2 and write the relation

$$\pi_1 = f(\pi_2),$$

then

$$\frac{\rho v^2 l^2}{R} = f(\pi_2),$$

or

$$R = \rho v^2 l^2 \frac{1}{f(\pi_2)},$$

which we may write in terms of a different function $\phi(\pi_2)$ as

$$R = \rho v^2 l^2 \phi(\pi_2).$$

The function $\phi(\pi_2)$ is, of course, dimensionless since π_2 is. Information regarding $\phi(\pi_2)$ can be obtained by experiments on a model or the full-sized machine. One of the valuable uses of dimensional analysis, as previously mentioned, is the prediction of the action of a machine from the behavior of its model. This use is explained in § 55 on the principle of similitude.

Certain restrictions on the choice of values for two of the unknowns in Eqs. (210) are evident. If in the determination of both π_1 and π_2 one of the unknowns (say w) were taken to be zero, it would mean that μ would be eliminated from both π_1 and π_2 and hence from the relation

$$F(\pi_1, \pi_2) = 0.$$

Thus the result, by this choice of value for w, would be independent of μ and consequently wrong. There are evidently advantageous ways to assign values to two of the variables. In this problem, we are interested in eventually solving the equation for R. Hence in π_1, u is chosen to be (-1) in order that π_1 is simply solvable for R. Obviously, it is not desirable for R to enter into both π_1 and π_2, for then the equation

$$F(\pi_1, \pi_2) = 0$$

is not solvable for R. Consequently in π_2 take $u = 0$. Thus the choice of values for $n - m$ of the unknowns depends upon the nature of the problem. Other restrictions are pointed out in the following example.

EXAMPLE 2. Let it be required to determine the form of the expression for the velocity of sound in a gas. Suppose this velocity depends upon the density ρ, pressure p, and viscosity μ. If M, L, and T are chosen as the three primary quantities, the dimensions of the secondary quantities v, ρ, p, and μ are respectively LT^{-1}, ML^{-3}, $ML^{-1}T^{-2}$, $ML^{-1}T^{-1}$. Since there are four secondary quantities and three primary quantities there will be but one dimensionless product, π_1, which can be formed out of them. This product may be written

$$\pi_1 = v^w \rho^x p^y \mu^z,$$

or

$$[\pi_1] = \left(\frac{L}{T}\right)^w \left(\frac{M}{L^3}\right)^x \left(\frac{M}{LT^2}\right)^y \left(\frac{M}{LT}\right)^z.$$

In order that π be dimensionless, the sums of the exponents of L, M, and T must be zero. Thus

$$-3x - y - z + w = 0,$$

$$x + y + z = 0,$$

$$-2y - z - w = 0.$$

One of the variables may be assigned a value. Since we shall solve for v, let $w = 1$. Then

$$x = \tfrac{1}{2},\ y = -\tfrac{1}{2},\ z = 0,$$

or

$$\pi_1 = v\rho^{1/2}p^{-1/2}.$$

Since

$$F(\pi_1) = 0$$

for all values of π_1, π_1 must be equal to a constant. Thus

$$\pi_1 = v\rho^{1/2}p^{-1/2} = \text{constant}.$$

Solving for v we have

$$\text{constant}\sqrt{\frac{p}{\rho}}.$$

The velocity apparently does not depend upon μ. This is correct. Let us note a further restriction on the choice of one of the variables w, x, y, z. Suppose z had been chosen equal to 1. If there is a solution

of the equations for x, y, and w the result will contain μ. But on assigning z the value 1 and attempting to solve for x, y, and w the determinant is seen to be

$$\begin{vmatrix} -3 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & -2 & -1 \end{vmatrix} = 0.$$

By § 26, there is no solution of the linear system. Consequently the error of supposing the result depends upon the viscosity is avoided.

EXAMPLE 3. Consider the familiar mass and spring system provided with damping vanes, as shown in Fig. 2. By dimensional analysis, let us obtain an expression for the steady-state amplitude of vibration A under the action of the vertical force $F \sin \omega t$. It will be seen that the result agrees, as far as it goes, with that obtained by solving the differential equation of motion. It is known that the following parameters are involved in the differential equation of motion of an oscillating mass:

PARAMETER	SYMBOL	DIMENSIONS
Angular frequency of applied force	ω	T^{-1}
Mass of body..................	M	M
Damping constant.	k_d	MT^{-1}
Peak value of applied force.....	F	MLT^{-2}
Amplitude of vibration.........	A	L
Spring constant................	k	MT^{-2}

There are six secondary quantities and three primary quantities. Consequently by the π theorem there are three dimensionless products. The values of π_1, π_2, π_3 are obtained as follows. Let π denote any one of the π's. Then

$$\pi = A^x \omega^y M^z k_d^u F^v k^w,$$

or

$$[\pi] = L^x (T^{-1})^y M^z (MT^{-1})^u (MLT^{-2})^v (MT^{-2})^w.$$

For this relation to be true, it is necessary that the sums of the exponents of L, M, and T to be zero, that is

$$x + v = 0,$$

$$z + u + v + w = 0,$$

$$y + u + 2v + 2w = 0.$$

Three values of the unknowns may be assigned arbitrarily.

If

$$y = z = 0, \quad x = 1,$$

then

$$v = -1, \quad u = 0, \quad w = 1,$$

and

$$\pi_1 = \frac{Ak}{F}.$$

If

$$x = v = 0, \quad w = 1,$$

then

$$y = -2, \quad z = -1, \quad u = 0,$$

and

$$\pi_2 = \frac{k}{M\omega^2}.$$

Finally if

$$x = z = 0, \quad u = 1,$$

then

$$y = 1, \quad v = 0, \quad w = -1,$$

and

$$\pi_3 = \frac{\omega k_d}{k}.$$

In accordance with the π theorem

$$F(\pi_1, \pi_2, \pi_3) = 0,$$

or

$$\pi_1 = \phi(\pi_2, \pi_3).$$

Then

$$\frac{Ak}{F} = \phi\left(\frac{k}{M\omega^2}, \frac{\omega k_d}{k}\right),$$

or

$$A = \frac{F}{k}\phi\left(\frac{k}{M\omega^2}, \frac{\omega k_d}{k}\right).$$

The unknown function ϕ, in this exceptional case, can be determined by the solution of the differential equation of motion. It is

$$\phi\left(\frac{k}{M\omega^2}, \frac{\omega k_d}{k}\right) = \frac{1}{\left[\left(1 - \frac{M\omega^2}{k}\right)^2 + \frac{k_d{}^2\omega^2}{k^2}\right]^{1/2}},$$

or

$$\frac{\phi(\pi_1, \pi_2)}{\left[\left(1 - \frac{1}{\pi_2}\right)^2 + \pi_3{}^2\right]^{\frac{1}{2}}}$$

EXAMPLE 4. *Unfortunate choice of primary quantities.* Let it be required to determine by dimensional analysis the stiffness of a beam (the ratio of load to deflection) as a function of the geometrical dimensions and moduli of elasticity. The parameters involved are:

PARAMETER	SYMBOL	DIMENSIONS
Stiffness........................	S	MT^{-2}
Length........................	l	L
Breadth........................	b	L
Depth........................	d	L
Young's modulus...............	E	$ML^{-1}T^{-2}$
Shear modulus	μ	$ML^{-1}T^{-2}$

Since there are six secondary and apparently three primary quantities entering the problem, three dimensionless π's are expected. We have

$$\pi = s^x l^y b^z d^u E^v \mu^w,$$

$$[\pi] = (MT^{-2})^x L^y L^z L^u (ML^{-1}T^{-2})^v (ML^{-1}T^{-2})^w.$$

It is necessary that

$$\begin{aligned} 0 + y + z + u - v - w &= 0, \\ x + 0 + 0 + 0 + v + w &= 0, \\ -2x + 0 + 0 + 0 - 2v - 2w &= 0. \end{aligned} \tag{211}$$

When any three of the unknowns are assigned values arbitrarily, the determinant of the coefficients of the remaining three unknowns is zero since every third-order determinant formed from the array of numbers

$$\begin{pmatrix} 0 & 1 & 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ -2 & 0 & 0 & 0 & -2 & -2 \end{pmatrix}$$

(called a **matrix**) is zero. This matrix is called the matrix of the coefficients of Eq. (211). In § 26 the rank of a determinant is defined. We now need the notion of the rank of a matrix. A matrix A is

said to have *rank* r if all determinants which can be formed from the rows and columns of A, of order greater than r, are zero while at least one r-rowed determinant is not zero.

Now Eqs. (211) have solutions. (Evidently $x = 1, y = -2, z = 0, u = 1, v = 1, w = -2$ is a solution.) In fact, we have the theorem: Any system of m homogeneous linear equations in n unknowns whose matrix is of rank $r < n$ has $n - r$ linearly independent solutions.[14] The rank of the matrix of (211) is 2. Hence there are $6 - 2 = 4$ independent solutions, and hence four π's. But originally we anticipated three π's!

By inspection of the dimensions of the six quantities, it is seen that M and T enter only in the combination MT^{-2}. Moreover, the system is static, and one is led to expect that all the secondary quantities are expressible in terms of force and length. The number of primary quantities chosen was too large.

If the solution is carried out using as primary quantities L and F, there are four π's and the final result is

$$S = El\phi\left(\frac{b}{l}, \frac{d}{l}, \frac{\mu}{E}\right).$$

EXAMPLE 5. *Use of a dimensional constant.* As a final example, consider a problem involving a dimensional constant. Let it be required to find, by dimensional methods, an expression for the period of revolution of a planet revolving about the sun. It is supposed that the period depends upon the mass of the sun S, the mass of the planet m, the distance d between their centers, and the gravitational constant G. Take as secondary quantities $S + m$, d, G, and period t. Take as primary quantities L, M, and T. One π is expected. Consequently

$$\pi = (S + m)^x d^y G^z t^u,$$

$$[\pi] = M^x L^y (M^{-1}L^3T^{-2})^z T^u.$$

Hence

$$x - z = 0,$$

$$y + 3z = 0,$$

$$-2z + u = 0.$$

Let $u = 1$, then $z = \frac{1}{2}$, $x = \frac{1}{2}$, $y = -\frac{3}{2}$,

$$F(\pi_1) = 0,$$

[14] See Refs. 17–18 at the end of the text.

for all values of π_1. Hence π_1 = constant. We now have

$$(S + m)^{1/2} d^{-3/2} G^{1/2} t^1 = \text{constant},$$

or

$$t = \text{constant} \frac{d^{3/2}}{G^{1/2}(S + m)^{1/2}}.$$

The value obtained by solving the differential equations of motion is

$$t = \frac{2\pi d^{3/2}}{G^{1/2}\sqrt{S + m}}.$$

The question may arise, when does a dimensional constant enter a problem? The answer is, it enters whenever the general equations underlying the phenomena considered contain a dimensional constant. The universal law of gravitation seldom is used in engineering. Consequently, we are not interested in this particular dimensional constant.

In electrodynamic problems involving the Maxwell field equations, the dimensional constant C (the velocity of light) enters.

Summary. It was pointed out in the introductory paragraphs of this section that very little mathematical knowledge was required in dimensional analysis. But it is now evident that in order to apply dimensional analysis throughout the whole fields of physics and engineering it is necessary to know the underlying equations and principles of these subjects. However, dimensional analysis is applied to only one problem at a time, and in any particular problem the dimensional constants and the secondary physical quantities of the problem must be known.

The arbitrary choice of unknowns in the homogeneous system of m equations in the application of the π theorem is governed not only by mathematics but also by physics and engineering. The mathematical theory is stated concisely by Dickson in the single theorem, "Given m homogeneous linear equations in n unknowns whose coefficients belong to any field F and have a matrix of rank r, we may select r of the equations so that their matrix has a non-vanishing r-rowed determinant. These r equations determine uniquely r of the unknowns as homogeneous linear functions, with coefficients in F, of the remaining $n - r$ unknowns. For all the values of the latter, the expressions for the r unknowns satisfy the given m equations."* The system of Eqs. (210) is of rank three. System of Eqs. (211) is of rank two. The theorem tells us which unknowns may be assigned arbitrarily. In Eqs. (211), the determinant $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ is not zero. Consequently,

* Dickson, *Modern Algebraic Theories*, Benj. H. Sanborn & Co.

u, v, w, and z may be assigned arbitrary values. The fact that the determinant $\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$ prevents the assigning of arbitrary values to x, u, v, and w.

The choice of unknowns depends also upon physics. It is implied in the π theorem that the matrix of the m equations is of rank m, not $r < m$. Hence the proper fundamental quantities must be chosen. In example 2, § 54, a doubtful secondary quantity was forced to remain temporarily in the result by assigning the exponent unity to it.

The choice of unknowns depends also upon engineering. The results must be in a form to avoid unnecessary expense in experimenting. The secondary quantity for which $F(\pi_1, \pi_2, \ldots) = 0$, is to be solved must be simply involved in exactly one π. In experiments using a model § 55, certain controllable secondary quantities must appear advantageously in the arguments of the unknown function ϕ.

This separation of theory is for explanatory purposes only; of course, no logical division is implied.

55. Principle of Similitude. The principle of similitude is, "The fundamental entities out of which the physical universe is constructed are of such a nature that from them a miniature universe could be constructed similar in every respect to the present universe." (See Ref. 29 at end of text.) Some physicists hold this principle to be an axiom of physics to be accepted without proof. Other physicists affirm it to be a natural consequence of the π theorem. Its importance to the engineer is that it asserts that, as far as the principles of physics are concerned, a model can always be constructed. The Buckingham π theorem furnishes the machinery by which we can obtain information regarding the action of the full-sized machine from the model. This has already been indicated for ships in result 12, § 49. Further examples are now considered.

EXAMPLE 1. The thrust T of a screw propeller of given shape depends upon the diameter D; the rate of revolution n; the speed of advance S; the density ρ, the kinematic viscosity ν of the water; and the acceleration of gravity g.

If L, M, and T are taken as primary quantities, the dimensions of thrust (force) are MLT^{-2}, and the dimensions of the other quantities are respectively (L), (T^{-1}), (LT^{-1}), (ML^{-3}), (L^2T^{-1}), and (LT^{-2}). Four dimensionless products may be formed. By the theory of § 54 the final result for the full-sized propeller is

$$T = \rho D^2 S^2 \phi\left(\frac{Dn}{S}, \frac{DS}{\nu}, \frac{Dg}{S^2}\right). \tag{212}$$

Likewise for a model

$$T' = \rho' D'^2 S'^2 \phi\left(\frac{D'n'}{S'}, \frac{D'S'}{\nu'}, \frac{D'g'}{S'^2}\right).$$

The propeller is said to be **physically similar** to its model if

$$\phi\left(\frac{Dn}{S}, \frac{DS}{\nu}, \frac{Dg}{S^2}\right) = \phi\left(\frac{D'n'}{S'}, \frac{D'S'}{\nu'}, \frac{D'g'}{S'^2}\right).$$

This equation is satisfied if

$$\left.\begin{aligned} \frac{Dn}{S} &= \frac{D'n'}{S'}, \\ \frac{DS}{\nu} &= \frac{D'S'}{\nu'}, \\ \frac{Dg}{S^2} &= \frac{D'g'}{S'^2}. \end{aligned}\right\} \tag{213}$$

The first of (213) holds if the ratio of tip speed of blades to speed of advance is the same in both the machine and model. If both are run in water $\nu = \nu'$, and of course $g = g'$. The last two equations then are

$$DS = D'S',$$

$$\frac{D^2}{S^2} = \frac{D'^2}{S'^2}.$$

These two simultaneous equations reduce to the conditions $D = D'$ and $S = S'$. Thus in size the machine is identical to the model and must be run at the same speed!

However, an approximation can be obtained. If the flow about the propeller is turbulent (and it is known to be so) then viscosity has little effect. Consequently we omit the second equation of (213). The last equation of (213) then gives

$$\frac{D}{D'} = \left(\frac{S}{S'}\right)^2.$$

The speeds S and S' are called **corresponding** speeds. If the last relation is substituted in

$$\frac{T}{T'} = \frac{\rho D^2 S^2 \phi\left(\frac{Dn}{S}, \frac{DS}{\nu}, \frac{Dg}{S^2}\right)}{\rho' D'^2 S'^2 \phi\left(\frac{D'n'}{S'}, \frac{D'S'}{\nu'}, \frac{D'g'}{S'^2}\right)},$$

ϕ cancels and the important result is obtained that

$$\frac{T}{T'} = \left(\frac{D}{D'}\right)^3.$$

Thus at *corresponding speeds*, the ratio of thrusts from propeller and model is the cube of the ratio of their diameters.

EXAMPLE 2. It is desired to predict the windage loss of a 600-r.p.m. synchronous condenser which has a rotor 96 in. in diameter and 90 in. long and which is to operate in hydrogen at atmospheric pressure. A careful determination of the windage loss as a function of speed (see Fig. 25) has been made on another synchronous condenser (hereafter called the model) of very similar design and construction but different size and rating. The model has a closed cooling system, but uses air instead of hydrogen as the cooling medium. It has a rotor 64 in. in diameter and 60 in. long. Under normal operating conditions, the average temperature of the cooling air is 35° C., and it is expected that the hydrogen in the other machine will run at about the same temperature. The coefficients of viscosity at 35° C. of air and hydrogen are respectively $4.05 \times 10^{-7} \frac{\text{slug}}{}$ and $2.1 \times 10^{-7} \frac{\text{slug}}{}$ The densities at 32° F. and atmospheric pressure of air and hydrogen are respectively 8.09×10^{-2} lb. per cu. ft. and 5.61×10^{-3} lb. per cu. ft.

Find, as accurately as possible, the windage loss at rated speed in the hydrogen-cooled machine from the test on the air-cooled "model."

It is assumed that the windage loss W depends on the following parameters:

PARAMETER	SYMBOL	DIMENSIONS
Diameter of rotor	D	L
Physical dimensions (given by ratio)	LD^{-1}	1
Speed	ω	T^{-1}
Density of cooling medium	ρ	ML^{-3}
Viscosity of cooling medium	μ	$ML^{-1}T^{-1}$

The dimensions of windage loss are ML^2T^{-3}. By the π theorem, there are three dimensionless products. Carrying out the solution according to the theory of the last section we have:

$$\pi = D^x\left(\frac{L}{D}\right)^y \omega^s \rho^u \mu^v W^w,$$

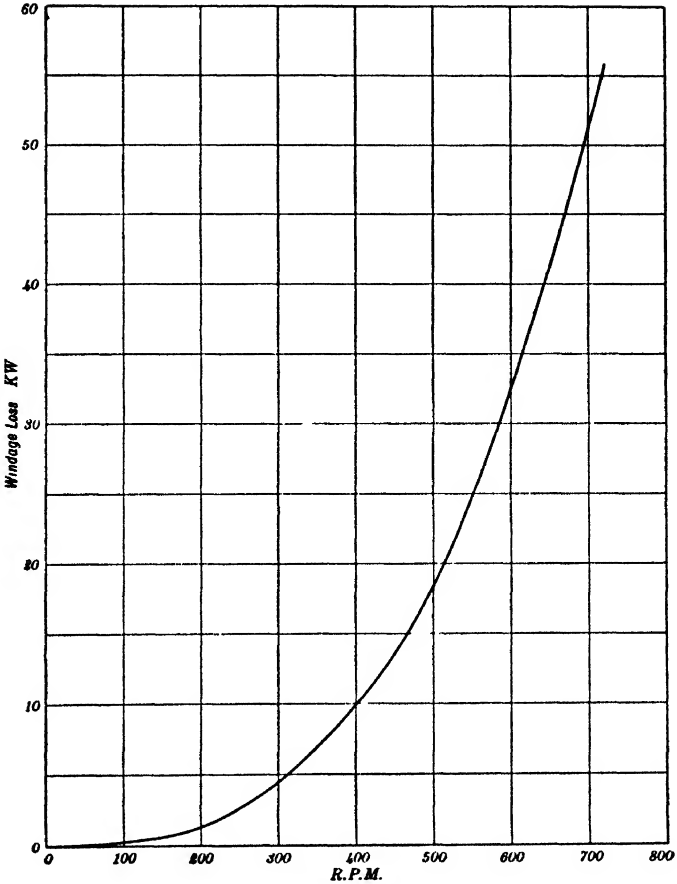


FIG. 25.—Windage Loss Synchronous Condenser.

or

$$[\pi] = L^x(1)^y(T^{-1})^z(ML^{-3})^u(ML^{-1}T^{-1})^v(ML^2T^{-3})^w$$

The linear system of equations is

$$x - 3u - v + 2w = 0,$$

$$u + v + w = 0,$$

$$-z - v - 3w = 0.$$

If

$$x = u = v = w = z = 0, \quad y = 1,$$

then

$$\pi_1 = \frac{L}{D}.$$

If

$$w = 1,\ u = 0,$$

then

$$v = -1,\ z = -2,\ x = -3,$$

and

$$\pi_2 = \frac{W}{\omega^2 D^3 \mu}.$$

Now W must not appear in π_3 since W is already in π_2 and since the function $F(\pi_1, \pi_2, \pi_3) = 0$ is to be solved for W. Since ω is the controllable factor entering into the functional relation shown in Fig. 25, ω must appear in π_3. Accordingly let

$$w = 0,\ z = 1.$$

Then

$$v = -1,\ u = 1,\ x = 2$$

and

$$\pi_3 = \frac{D^2\omega\rho}{\mu}.$$

By the π theorem, for the machine and model, we have respectively

$$W = D^3\mu\omega^2\phi(\pi_1, \pi_3) = D^3\mu\omega^2\phi\left(\frac{L}{D}, \frac{D^2\omega\rho}{\mu}\right),$$

$$W' = D'^3\mu'\omega'^2\phi(\pi'_1, \pi'_3) = D'^3\mu'\omega'^2\phi\left(\frac{L'}{D'}, \frac{D'^2\omega'\rho'}{\mu'}\right).$$

The function ϕ will be eliminated from the ratio $\frac{W}{W'}$ if $\frac{L}{D} = \frac{L'}{D'}$ and $\frac{D^2\omega\rho}{\mu} = \frac{D'^2\omega'\rho'}{\mu'}$. The first equation is evidently satisfied. Solving the last equation for ω'

$$\omega' \text{ (“corresponding” speed of model)} = \left(\frac{D}{D'}\right)^2\left(\frac{\rho}{\rho'}\right)\left(\frac{\mu'}{\mu}\right)\omega$$

$$= (1.5)^2\left(\frac{5.61}{80.9}\right)\left(\frac{4.05}{2.1}\right)600 = 180.5 \text{ r.p.m.}$$

Since

$$\frac{W}{W'} = \left(\frac{D}{D'}\right)^3 \left(\frac{\mu}{\mu'}\right) \left(\frac{\omega}{\omega'}\right)^2,$$

$$W = \frac{(1.5)^3(2.1)\ (600)^2}{(4.05)\ (180.5)^2} = 19.3W'.$$

But from Fig. 25, $W' = 1$ kw. at the "corresponding" speed, 180.5 r.p.m. Hence, the expected windage loss in the hydrogen machine at rated speed is 19.3 kw.

56. Systematic Experimentation. The value of dimensional analysis in experimental work is evident even in the simplest physical systems. For example, suppose the time t of swing of a rotor is investigated (result 3, § 49) by unsystematic experimentation. It is known that t depends upon I, T_s, and ϕ_0. By varying all parameters, one and then two at a time, curves are obtainable which give t as a function of these arguments. But if, by dimensional analysis, the relation $t = \sqrt{\frac{I}{T_s}}\,F(\phi_0)$ is obtained, it is necessary to determine by experiment only $F(\phi_0)$, which involves varying only ϕ_0. If the desired quantity is a function of many arguments, it is at once evident that the saving made by the use of dimensional analysis is enormous.

57. An Additional Method. In the prediction of results by use of a model, the method indicated in the following example is sometimes of use either independently or in conjunction with the π theorem. It is especially applicable if the general partial differential equations defining the phenomenon are known but cannot be integrated, as in the study of heat convection. Obviously, it is not feasible to discuss the problems involving partial differential equations at this point, but an indication of the method can be given by considering a trivial example with an ordinary differential equation.

Suppose that we have a physical system for which the differential equation $m\dfrac{d^2x}{dt^2} = f$ is valid. In this equation, we have mass, acceleration, and force. Suppose that we are ignorant of the dimensions of force. The problem is to find on what fundamental or primary quantities force depends. The differential equation for the model is $m_1\dfrac{d^2x_1}{dt_1{}^2} = f_1$. The model is similar to the machine and hence $m = Mm_1$, $x = Lx_1$, $t = Tt_1$, and $f = Ff_1$, where M, L, T, and F are numbers. M, L, and T are known, and we wish to find F. Substituting in

$m\frac{d^2x}{dt^2} = f$ the values of x, m, t, and f in terms of x_1, m_1, t_1, and f_1, we have

$$m_1\frac{d^2x_1}{dt_1^2} = \frac{FT^2}{LM}f_1.$$

Comparing the last equation with

$$m_1\frac{d^2x_1}{dt_1^2} = f_1$$

we have

$\frac{FT^2}{LM} = 1$ (a dimensionless product of the quantities and hence a π),

or

$$F = \frac{ML}{T^2}.$$

Now suppose m_1, x_1, t_1 represent units of mass, length, and time. The Mm_1, Lx_1, and Tt_1 represent the number of units of each in the machine. The final result states that

$$F = \frac{\text{(A number representing units of mass) (A number representing length)}}{\text{(A number representing time)}^2}$$

and consequently the dimension of force are $\frac{\text{(Mass} \times \text{length)}}{\text{(Time)}^2}$ By a very similar method, Nusselt (Ref. 30 at end of text), by a change of variables in the partial differential equations of machine and model, has determined relations which must hold between units in the heat-transfer equations. These relations give, in effect, the π's of the π theorem. The method has the advantage that it sometimes gives a smaller number of π's than the π theorem and consequently a more useful result.

58. Summary. The processes of this section are now summarized.

(a) If an engineering equation is known to be dimensionally homogenous it may be checked for dimensions as indicated in § 50.

(b) A change of units of the first kind is accomplished by the rule of § 51, part (a). A change of units of the second kind is carried out by mere routine substitution in Eqs. (207) and (209).

(c) The steps in carrying out a solution by the π theorem are as follows:

1. Decide on what physical quantities and dimensional constants the unknown quantity depends. This decision rests upon general knowledge of the physical field in which the problem lies.

2. Select the proper primary quantities and form the table of parameters as illustrated in example 3, § 54.

3. Write an expression for a general π. Write the dimensional formula for this π in terms of the primary quantities. Form the homogeneous linear system of algebraic equations.

4. Assign values to $n - m$ of the unknowns in accordance with the theory in the last paragraphs of § 54.

5. Solve for the unknown physical quantity as indicated by the equation

$$\alpha = \beta^{k_1} \gamma^{k_2} \ldots \phi(\pi_2, \pi_3, \ldots).$$

(d) To investigate the behavior of a machine from a model obtain, by the π theorem the two equations:

$$\alpha = \beta^{k_1} \gamma^{k_2} \ldots \phi(\pi_2, \pi_3 \ldots) \quad \text{(for the machine)},$$

$$\alpha' = \beta'^{k_1} \gamma'^{k_2} \ldots \phi(\pi'_2, \pi'_3, \ldots) \quad \text{(for the model)}.$$

If the conditions of physical similarity are satisfied, i.e., if $\pi_i = \pi'_i$ $(i = 2, 3, \ldots n - m)$, the ratio α/α' does not contain the unknown function, and $\alpha = \alpha' \left(\frac{\beta}{\beta'}\right)^{k_1} \left(\frac{\gamma}{\gamma'}\right)^{k_2} \ldots$.

PROBLEMS

1. Obtain, by dimensional analysis, all the results of § 49, which have not been solved as illustrative examples in this section.

2. It is desired to design a 1750-r.p.m. centrifugal pump which will deliver 2,600,000 lb. per hr. of mercury against a head of 85 ft. of mercury. As a first step in the design, a standard water pump is to be chosen which will give the desired performance when used with mercury. The catalogue data for water pumps include the following items for each pump:

(*a*) operating speed S_1,
(*b*) head of water pumped against h_1,
(*c*) volume of water delivered in unit time Q_1.

In order to choose the correct pump to be used to pump the liquid mercury, relations between S_1, h_1, and Q_1 may be obtained which must be fulfilled by the water pump which will give the desired performance with mercury.

Considering geometrically similar pumps, the important factors are:

(*a*) a characteristic dimension, such as the impeller diameter,
(*b*) quantity of fluid to be pumped per unit time,
(*c*) the pressure head to be pumped against,

(*d*) the speed at which the pump is run,
(*e*) the density of the fluid,
(*f*) possibly, the viscosity of the fluid.

Assuming that viscosity does not play an important part in the pumping process, obtain expressions for or relations between Q_1, h_1, S_1 for a water pump that will give the desired performance when used with mercury.

Repeat, taking viscosity of the fluid pumped into account.

For water at 20° C.	For mercury at 20° C.
Density = 1 gram per cc.	Density = 13.6 grams per cc.
Viscosity $= 0.0101 \dfrac{\text{grams}}{\text{cm. sec.}}$	Viscosity $= 0.0159 \dfrac{\text{grams}}{\text{cm. sec.}}$

VI

GRAPHICAL AND NUMERICAL METHODS OF SOLVING DIFFERENTIAL EQUATIONS

Thus far, the elementary principles of the first part of the present chapter have led, in general, to a linear differential equation, or to a system of such equations, with constant coefficients. The integration of such equations has meant expressing the solution in the form of a sum of a finite number of elementary functions. The same principles and others also lead to equations whose solutions cannot be so expressed. Such equations need not be linear, and the coefficients instead of being constants may involve both the dependent and independent variables. The integration of such equations, more often than not is very difficult, and their analytic solution is reserved for Vol. II, Chap. II. At present, numerical solutions may be obtained without additional mathematical knowledge.

59. Nature of Numerical Integration. The solution obtained by graphical or numerical integration of a differential equation is a graph of the function which satisfies the differential equation and the initial conditions. For example, the numerical solution of

$$L\frac{di}{dt} + Ri = E, \tag{214}$$

where E, R, and L have the numerical values 100, 20, and 10, and i is zero at $t = 0$, is shown in Fig. 26. The curve shown is the solution of (214) only for the above values of E, R, and L. If different values are assigned, all work must be repeated and this is one of the great disadvantages of numerical integration. We thus say that in numerical integration the parameters are lost from the solution.

There exists a general process by which it is always possible to obtain the solution (if it exists) of a single differential equation or of a system of such equations. This general process is usually tedious to apply and is employed only as a last resort. For particular problems of frequent occurrence, special methods are available which yield the desired result more easily. Accordingly, these methods are given first.

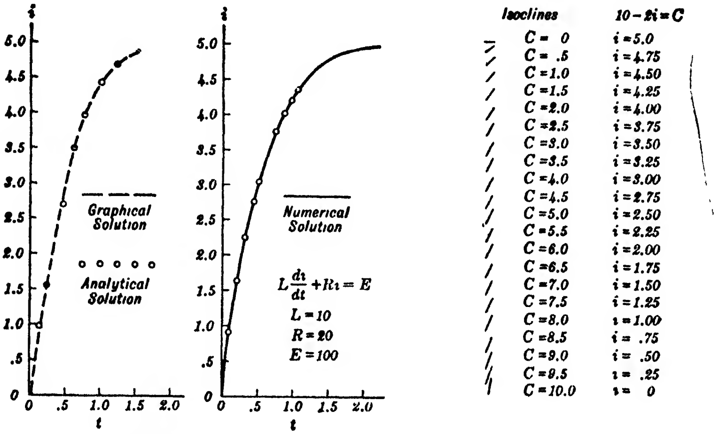


FIG. 26.—Numerical and Graphical Integration of Differential Equations.

60. The Differential Equation $\frac{di}{dt} = f(i;\, t)$. The graphical and numerical methods in this case are very simple. They are based on the fact that $\frac{di}{dt}$ is the slope, at the point (t, i), of the curve which is the solution of the differential equation. Consequently, if at a sufficient number of points (t, i) the values of $f(i;\, t)$ be computed the direction elements may be drawn at each of these points. If enough direction elements be put in, the directions of the solutions at any point are given and the curves may be sketched which are the infinitude of solutions of the differential equation. Among these curves there will be one which satisfies the initial condition $i = i_0(t_0)$, and is the solution of the physical problem considered.

Graphical solution. A convenient way of carrying out a graphical solution is known as the method of isoclines. A curve obtained by setting $f(i;\, t) = C$ (a constant) is called an isocline because at every point of such a curve the slope $\frac{di}{dt} = f(i;\, t) = C$ is a constant. Let the

family of curves be drawn for various assigned values of C. For instance, at every point on the isocline $f(i; t) = -1$ (or c_1) the tangent makes an angle of 135° (or arc $\tan^{-1} c_1$) with the positive x-axis. By inserting a sufficiently large number of isoclines the tangents to the solution and hence the solutions themselves may be drawn.

EXAMPLE 1. Let it be required to obtain the graphical solution of (214), i.e., $\frac{di}{dt} = \frac{E - Ri}{L} = 10 - 2i$, for the initial condition $i = 0$ for $t = 0$.

The isoclines $10 - 2i = C(C = 0, .5, 1, 1.5, 2, \ldots, 9.5, 10)$ are lines parallel to the t-axis (Fig. 26). The third column from the right indicates the slopes at which the directional elements cut the isoclines. Beginning at the origin and inserting successively the directional elements whose slopes are 10, 9.5, 9, 8.5, 8, . . ., 0, we have the curve composed of arrows which are tangents to the solution of the differential equation. The solution itself is now easily drawn. The correct solution, found by analytic integration, passes through the centers of the small circles shown on the broken line curve.

Numerical solution. A convenient way of carrying out a numerical solution of $\frac{di}{dt} = f(i; t)$ is as follows. Suppose $i = \phi(t)$ is the solution of $\frac{di}{dt} = f(i; t)$ satisfying the initial condition $i = i_0(t_0)$. Let $t_0, t_1, t_2, \ldots$ and i_0, i_1, i_2 denote the quantities shown in Fig. 27. Let

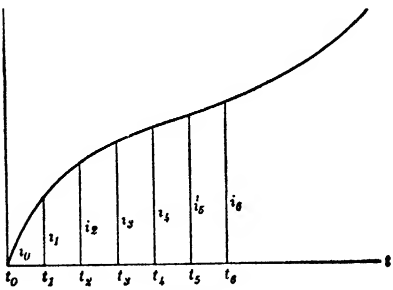


FIG. 27.

$t_{k+1} - t_k = h$ $(k = 0, 1, 2, \ldots)$. If h and the curvature of $i = \phi(t)$ are small, then the value of $\frac{d\phi}{dt}$ within the interval h is given approximately by

$$\left.\frac{d\phi}{dt}\right| \qquad \phi_{k+1} - \phi_k$$

From this equation

$$\phi_{k+1} \doteq \phi_k + h\frac{d\phi}{dt}\bigg|_{\text{ave.}}.$$

Let the average value $\frac{d\phi}{dt}\Big|_{\text{ave.}}$ be approximated by

$$\frac{d\phi}{dt}\bigg|_{\text{ave.}} \doteq \frac{\frac{d\phi}{dt}\Big|_k + \frac{d\phi}{dt}\Big|_{k+1}}{2},$$

where $\frac{d\phi}{dt}\Big|$ and $\frac{d\phi}{dt}\Big|_{k+1}$ are respectively the slopes of $i = \phi(t)$ at the points (t_k, i_k) and (t_{k+1}, i_{k+1}). Thus

$$\phi_{k+1} \doteq \phi_k + \frac{h}{2}\left(\frac{d\phi}{dt}_k + \frac{d\phi}{dt}\bigg|_{k+1}\right). \tag{215}$$

In constructing the solution, the value $i_0 = \phi_0$ is known from the initial condition and $\frac{d\phi}{dt}$ is easily computed from the differential equation. The value of $\frac{d\phi}{dt}$ is estimated from the slope of the tangent at (t_0, i_0). The value of ϕ_1 is then computed from (215) by letting $k = 0$. If this value of ϕ_1 when substituted for i in the right member of $\frac{di}{dt} = f(i; t)$ renders $\frac{di}{dt} \quad \frac{d\phi}{dt}$, then the value of ϕ_1 is sufficiently accurate and the point (t_1, i_1) has been located on the curve. Next $\frac{d\phi}{dt}\Big|$ is estimated and ϕ_2 calculated by (215). ϕ_2 is checked in the differential equation if $\frac{d\phi}{dt}$ The process is continued for $i_3, i_4, \ldots$

EXAMPLE 2. Let us solve, by the method just given, $\frac{di}{dt} = 10 - 2i$ subject to the initial condition $i = 0$ at $t = 0$. Then,

$$\phi_0 = 0, \quad \frac{di}{dt} \quad \frac{d\phi}{dt} \quad 10.$$

If $h = 0.1$ by (215),

$$\phi_1 = \phi_0 + \frac{0.1}{2}\left(10 + \frac{d\phi}{dt}\bigg|_1\right).$$

From the slope of the tangent at $t = 0$, estimate $\left.\dfrac{d\phi}{dt}\right|_1 = 9$. Then $\phi_1 = 0.95$. Substituting $\phi_1 = 0.95$ in

$$\frac{di}{dt} = 10 - 2\phi, \quad \left.\frac{di}{dt}\right|_1 = 8.1.$$

The estimated slope is too large. Try $\left.\dfrac{d\phi}{dt}\right|_1 = 8.2$. Then $\phi_1 = 0.91$ and $\left.\dfrac{di}{dt}\right|_1 = 8.2$. Thus

$$\left.\frac{di}{dt}\right|_1 = \left.\frac{d\phi}{dt}\right|_1 = 8.18.$$

Next

$$\phi_2 = \phi_1 + \tfrac{1}{2}\left(8.2 + \left.\frac{d\phi}{dt}\right|_2\right).$$

Continuing the process, we have the table:

t	Estimated $\left.\dfrac{d\phi}{dt}\right\|_{k+1}$	ϕ_{k+1}	$\left.\dfrac{di}{dt}\right\|_{k+1}$
0	10.0	....	10.00
0.1	8.2	0.91	8.18
0.2	6.7	1.65	6.70
0.3	5 5	2.26	5.48
0.4	4.5	2.76	4.50
0.5	3.7	3.16	3.68
0.6	3.0	3.49	3.02
0.7	2.5	3.77	2.46
0.8	2.0	3.99	2.02
0.9	1.6	4.17	1.66
1.0	1.3	4.32	1.36

The graph is the second curve in Fig. 26.

Example 3. Find the solution $E = \phi(t)$ of

$$\frac{dE}{dt} = \frac{e - ri}{K \times 10^{-8}} = \frac{e - rF(E)}{K \times 10^{-8}} \tag{216}$$

subject to the initial conditions $i = 0$ for $t = 0$ and where $i = F(E)$ is given, not by an equation, but by the curve of Fig. 28. This problem arises in determining the time rate of build-up of the armature voltage of a separately excited direct-current generator, when a constant voltage is applied to the field.

The symbols have the following significance:

E = armature voltage,
e = constant voltage applied to field,
i = field current,
r = resistance of the full circuit,
K = ratio of field flux linkages to armature voltage.

The constant K depends upon the construction of the generator, as well as the speed, which is assumed constant, and $i = F(E)$ is the saturation curve of the generator.

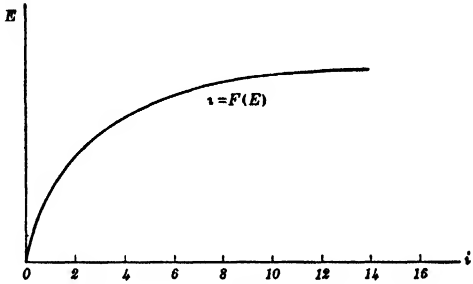


FIG. 28.

Assume values of 0, 1, 2, 3, . . . for i, read E from the curve $i = F(E)$, and compute $\frac{dE}{dt}$ for $i = 0, 1, 2, 3, \ldots$. This gives values of E and $\frac{dE}{dt}$ for a number of points. Since from physical considerations, out of which the problem arose, E and i are both continuous functions of t, we can obtain $\frac{dE}{dt}$ at points whose abscissas $t_0, t_1, t_2, \ldots$ are as near together as we please by taking $i_{k+1} - i_k = h$ sufficiently small where $k = 0, 1, 2, \ldots$.

Since $E = \phi(t)$ passes through the origin and we know its slope at $t_0, t_1, t_2, t_3, \ldots$ we have shown how to obtain its graph (Fig. 29) and the problem is theoretically solved.

Instead of plotting E as a function of time, a different procedure is sometimes followed. From Eq. (216), $\frac{dE}{dt}$ is determined as a function of E and then the reciprocal of $\frac{dE}{dt}$, or $\frac{dt}{dE}$, is plotted as the function of E. (See Fig. 30.)

Thus

$$\frac{dt}{dE} = \frac{K \times 10^{-8}}{e - rF(E)},$$

$$t = \int_0^E \frac{K \times 10^{-8}}{e - rF(E)} dE.$$

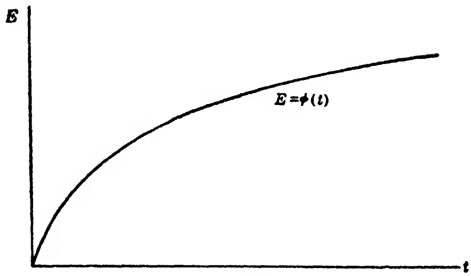


FIG. 29.

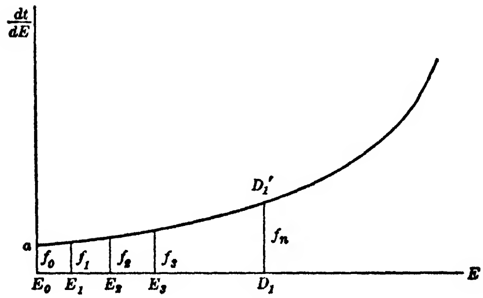


FIG. 30.

Evidently the time taken by the generator to attain the voltage $E = D_1$ is the area $E_0D_1D'_1a$ on the figure, and this may be found by numerical integration, for example by the trapezoidal formula

$$\int_a^b f(E)dE = h(\tfrac{1}{2}f_0 + f_1 + f_2 + \ldots + f_{n-1} + \tfrac{1}{2}f_n),$$

where

$$E_{k+1} - E_k = h \qquad (k = 0, 1, \ldots n - 1)$$

and f_k is the ordinate of the curve at $E = E_k$.

61. The System of Differential Equations:

$$\begin{aligned} \frac{dx}{dt} &= f(x,y), \\ \frac{dy}{dt} &= g(x,y). \end{aligned} \tag{217}$$

Eqs. (217) may be integrated numerically by the general method to be explained in §§ 63–65, or they may be easily integrated graphically in the following manner.

If the second of (217) is divided by the first there results

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} = h(x,y). \tag{218}$$

Eq. (218) is integrable by the method of isoclines, and the result is a functional relation (given by a graph) between x and y. Let this relation be denoted by $y = f_1(x)$ or $x = f_2(y)$. Suppose the initial conditions for the system (217) are $x_0 = x(t_0)$ and $y_0 = y(t_0)$. From (217), x and y can be expressed as functions of t by the relations,

$$\begin{aligned} \int_0^t dt &= \int_{x_0}^{x} \frac{dx}{f(x,y)} = \int_{x_0}^{x} \frac{dx}{f[x,f_1(x)]}, \\ \int_0^t dt &= \int_{y_0}^{y} \frac{dy}{g(x,y)} = \int_{y_0}^{y} \frac{dy}{g[f_2(y),y]}. \end{aligned} \tag{219}$$

The integrals in Eqs. (219) can be evaluated by the trapezoidal rule.

Example. Integrate $\frac{d^2y}{dt^2} = 0.9 - \sin y$ subject to the initial conditions $y = 0.925$ (radian) and $\frac{dy}{dt} = 0$ for $t = 0$. The differential equation may be replaced by the system

$$\frac{dx}{dt} = 0.9 - \sin y, \quad \frac{dy}{dt} \quad x. \tag{220}$$

Eqs. (220) are called the **normal form** of $\frac{d^2y}{dt^2} \quad 0.9 - \sin y$. The method of reducing any differential equation of the nth order, or any system of differential equations, to the normal form is explained in § 64.

To carry out the graphical solution of (220), divide the second by the first and obtain

$$\frac{dy}{dx} = \frac{0.9 - \sin y}{x}. \tag{221}$$

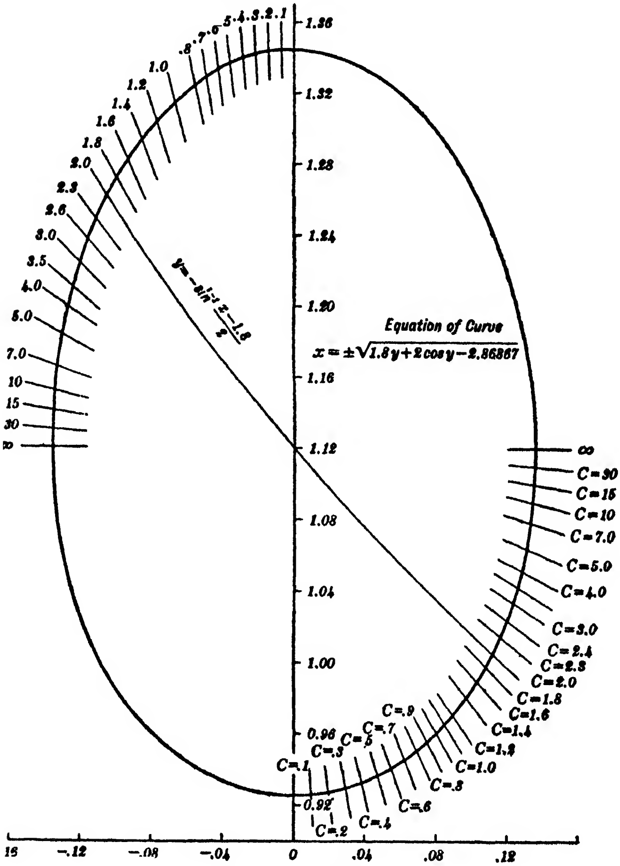


FIG. 31.—Method of Isoclines.

Eq. (221) corresponds to (218). The isoclines are

$$\frac{0.9 - \sin y}{x} = \frac{1}{C},$$

or

$$y = -\sin^{-1}\frac{1}{C}(x - 0.9C).$$

If C is assigned the values 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.5, 1.7, 2, . . . 100, we obtain the isoclines shown in Fig. 31. The slopes on the isoclines $x = 0$, $y = -\sin^{-1}\frac{1}{0.2}(x - 0.18)$, and $y = -\sin^{-1}\frac{1}{0.4}(x - 0.36)$ are respectively 0, 0.2, and 0.4. Continuing, we draw the oval curve in Fig. 31. In this particular problem, we can obtain the equation of the oval curve by integrating (221). This is unusual, however. The analytic solution of (221) satisfying the initial conditions of the problem is

$$x = \pm [1.8y + 2\cos y - 2.86867]^{1/2}.$$

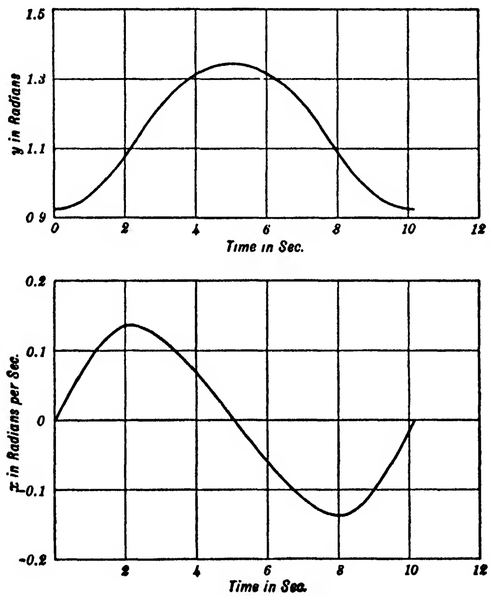


FIG. 32.—Results Simplified Hunting Equation.

The curves expressing x and y as functions of t are found by carrying out numerically, by the trapezoidal rule, the integrations indicated in Eqs. (219). From the second equation in (219)

$$= \int_{0.925}^{y} \frac{dy}{x}.$$

The values of x and y in this integral are the abscissa and ordinate of the point $P(x, y)$ on the oval curve in Fig. 31. The numerical integration is given in the following table.

y	x	$\frac{1}{x}$	$av\ \frac{1}{x}$	Δy	Δt	t
0.925	0					
0.9255	0.0101	99.01	99.01	0.0005	0.05	0.05
0.927	0.0201	49.75	74.38	0.0015	0.11	0.16
0.934	0.0421	23.75	36.75	0.007	0.26	0.42
0.942	0.0572	17.48	20.62	0 008	0.16	0.58
0.960	0.0798	12.53	15.00	0 018	0.27	0.85
0.995	0.1065	9.39	10.96	0.035	0.38	1.23
1.030	0.1223	8.18	8.78	0.035	0.31	1.54
1.065	0.1317	7.59	7 88	0.035	0.28	1.82
1.100	0.1361	7.35	7 47	0 035	0.26	2.08
1.135	0.1364	7.33	7 34	0.035	0.26	2.34
1.169	0 1329	7.52	7.42	0.034	0.25	2.59
1.204	0.1256	7.96	7 74	0.035	0.27	2.86
1.239	0.1141	8.76	8.36	0.035	0.29	3.15
1 274	0 0972	10.29	9 52	0.035	0.33	3 48
1.309	0 0718	13.93	12.11	0.035	0.42	3.90
1.320	0.0602	16.61	15 27	0 021	0.32	4.22
1.330	0 0479	20 88	18 74	0 010	0.19	4.41
1.340	0.0288	34.72	27.80	0 010	0.28	4.69
1.344	0 0156	64.10	49.41	0 004	0.20	4.89
1.345	0.006	166 67	115.38	0 001	0.12	5.01
1 345	−0 006	−166.67	−166.67	−0.0008	0.13	5.14
1 344	−0.0156	−64.10	−115.38	−0 001	0.12	5.26
1.340	−0 0288	−34.72	−49 41	−0 004	0.20	5.46
1.330	−0 0479	−20.88	−27.80	−0.010	0 28	5.74
1.320	−0.0602	−16 61	−18.74	−0.010	0.19	5.93
1 309	−0 0718	−13 93	−15 27	−0.021	0 32	6.25

The functional relations shown in this table between y and t and between x and t are shown also in Fig. 32.

62. The Radius of Curvature Method. Equations of the form,

$$\frac{d^2y}{dt^2} = f\left(y, \frac{dy}{dt}; t\right),$$

or

$$y'' = f(y, y'; t) \tag{222}$$

can be integrated graphically as follows. Dividing Eq. (222) by $(1 + y'^2)^{3/2}$, we have

$$\frac{y''}{(1 + y'^2)^{3/2}} = \frac{f(y, y'; t)}{(1 + y'^2)^{3/2}} \equiv g(y, y'; t).$$

The function $g(y, y'; t)$ is thus the curvature of the solution of Eq. (222). Consequently, the radius of curvature R is

$$R = \frac{1}{g(y, y'; t)}. \tag{223}$$

From the example of § 61, it is evident that through every point (t, y) of the ty-plane there passes an infinitude of solutions of a second-order differential equation. There is, however, only one solution having a prescribed direction through each point of the plane.

Beginning at a prescribed point P_0 in a prescribed direction $\left.\frac{dy}{dt}\right|_0 = D_0$ draw through P_0 a circle C_0 whose radius of curvature R_0 is computed by (223). (See Fig. 33.) At P_0, y, y', and t are all

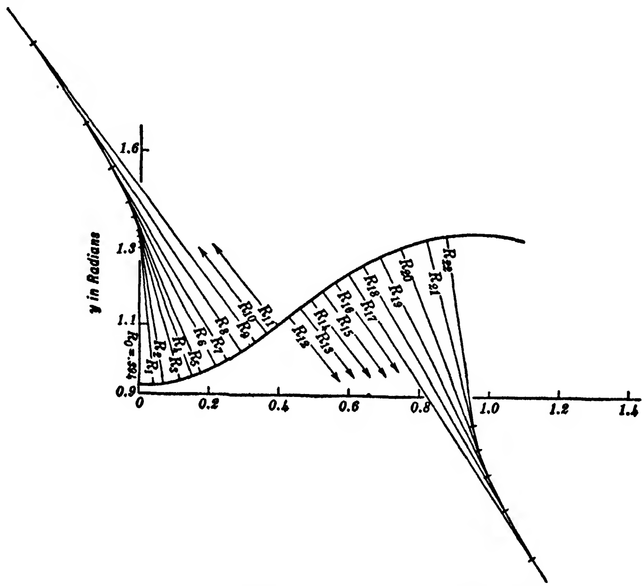


FIG. 33.—Integration by Curvature Method.

known from the initial conditions of the problem. At a point P_1 near P_0 and on the circle C_0 determine graphically y, y', and t. Substituting these values at P_1 in (223), compute the radius of curvature R_1 for the circle C_1 through P_1 and tangent to C_0 at P_1. Continuing this process, we obtain the points P_0, P_1, P_2, . . . on the solution of the differential equation. If the process is repeated by taking P_0P_1, P_2P_2 . . . equal to half their first values and the curve of the solution is but slightly changed the solution is sufficiently accurate.

EXAMPLE 1. In order that the different methods of this section can be easily compared, obtain the solution of the example of § 61 by the method of § 62.

To obtain convenient values of R for graphical work it is advantageous to make a change of independent variable in the equation

$$\frac{d^2y}{dt^2} = 0.9 - \sin y.$$

Accordingly, let $5\tau = t$. Then the differential equation to be solved graphically is

$$\frac{d^2y}{d\tau^2} = 25(0.9 - \sin y).$$

The radius of curvature at the point (τ, y) of the solution of the differential equation is

$$R = 0.04 \frac{\left[1 + \left(\frac{dy}{d\tau}\right)^2\right]^{3/2}}{0.9 - \sin y}$$

Let the coordinates of P be (0, 0.925). The values of y, y', and R at the points P_0, P_1, P_2 . . . are found, by the method of this article, to be as follows:

y	τ	y'	$R_i (i = 0, 1, \ldots, 22)$
0.925	0	0	0.39
0.927	0 046	0.080	0.40
0.930	0.072	0.175	0.42
0.940	0.120	0.260	0.48
0.950	0.153	0.350	0.55
0.960	0.181	0.420	0.63
0.980	0.220	0.500	0.80
1.000	0.255	0.570	1 03
1.025	0.295	0.690	1.57
1.050	0 337	0.750	2.39
1.075	0.368	0.790	4.04
1.100	0.400	0.800	9.54
1.125	0.431	0.800	−35.00
1.150	0.462	0.800	−6.62
1.175	0.495	0.790	−3.63
1.200	0.526	0.760	−2.47
1.225	0 562	0.700	−1.78
1.250	0 600	0.640	−1.37
1.275	0.640	0.550	−1.05
1.300	0.685	0.500	−0.88
1.325	0.744	0.410	−0.73
1.350	0.820	0.260	−0.58
1.360	0.880	0.150	−0.53

The construction is shown in Fig. 33. If the time is to be expressed in seconds the numbers along the τ-axis must be multiplied by 5.

63. Preliminary Ideas for the General Method of Numerical Integration. If the differential equation or the system of differential equations is not too complicated, the methods so far given will give a graphical or numerical solution. But systems are occasionally obtained where recourse must be had to the general method of numerical integration referred to in § 59. This general method is laborious and such that one significant error made at any step invalidates all the work following it. Consequently it is best to develop formulas, tables, and methods of procedure which reduce the labor to mere routine in order that full attention may be given to the actual numerical work. The use of these formulas and tables will become apparent in § 65.

We need first the equation of a polynomial which approximates a given smooth curve over a given interval. Suppose that the equation of the curve is $y = f(t)$ and that the curve passes through the points (t_n,y_n), (t_{n-1},y_{n-1}), (t_{n-2},y_{n-2}), . . . (t_1,y_1), where

$$t_{i+1} - t_i = h \qquad (i = n - 1, n - 2, \ldots, 1).$$

Let the approximating polynomial of the nth degree be written in the form

$$\begin{aligned} F(t) = a_0 + a_1(t - t_n) + a_2(t - t_n)(t - t_{n-1}) \\ + a_3(t - t_n)(t - t_{n-1})(t - t_{n-2}) + \ldots \\ + a_n(t - t_n)(t - t_{n-1}) \ldots (t - t_1). \end{aligned} \tag{224}$$

The coefficients a_0, a_1, . . . , a_n are determined by the conditions

$$F(t_n) = y_n, \quad F(t_{n-1}) = y_{n-1}, \ldots F(t_1) = y_1.$$

Applying these conditions to (224), we have

$$\begin{aligned} F(t_n) &= y_n = a_0, \\ F(t_{n-1}) &= y_{n-1} = a_0 + a_1(t_{n-1} - t_n), \\ F(t_{n-2}) &= y_{n-2} = a_0 + a_1(t_{n-2} - t_n) + a_2(t_{n-2} - t_n)(t_{n-2} - t_{n-1}), \\ &\text{etc.} \end{aligned}$$

From these equations,

$$\left.\begin{aligned} a_0 &= y_n, \\ a_1 &= \frac{y_n - y_{n-1}}{h}, \\ a_2 &= \frac{y_n - 2y_{n-1} + y_{n-2}}{2h^2}, \\ &\text{etc.} \end{aligned}\right\} \tag{225}$$

We need next **difference tables.** By means of differences, Eq. (225) can be more conveniently expressed. Suppose that the values of y (say $y_0, y_1, y_2, \ldots$) corresponding to the values $t = t_0, t_1, t_2, t_3, \ldots$ are known. Further let

$$t_{i+1} - t_i = h \quad \text{for} \quad i = 0, 1, 2, 3, \ldots.$$

Then the first differences of the function $y = f(t)$ (written $\Delta_1 y$) are defined to be

$$\begin{aligned} \Delta_1 y_1 &= y_1 - y_0, \\ \Delta_1 y_2 &= y_2 - y_1, \\ \Delta_1 y_3 &= y_3 - y_2, \\ &\text{etc.} \end{aligned}$$

The second differences are the first differences of the first differences:

$$\begin{aligned} \Delta_2 y_2 &= \Delta_1 y_2 - \Delta_1 y_1 = y_2 - 2y_1 + y_0, \\ \Delta_2 y_3 &= \Delta_1 y_3 - \Delta_1 y_2 = y_3 - 2y_2 + y_1, \\ &\text{etc.} \end{aligned}$$

Similarly, third and higher differences are formed. A difference table for the function $y = f(t)$ is written:

t	y	$\Delta_1 y$	$\Delta_2 y$	$\Delta_3 y$	$\Delta_4 y$
t_0	y_0				
t_1	y_1	$\Delta_1 y_1$			
t_2	y_2	$\Delta_1 y_2$	$\Delta_2 y_2$		
t_3	y_3	$\Delta_1 y_3$	$\Delta_2 y_3$	$\Delta_3 y_3$	
t_4	y_4	$\Delta_1 y_4$	$\Delta_2 y_4$	$\Delta_3 y_4$	$\Delta_4 y_4$
etc.					

To obtain from this table, say, $\Delta_2 y_4$, find in the same row the entry in the first column to the left, $\Delta_1 y_4$. From the last-mentioned element subtract the element above it, thus

$$\Delta_2 y_4 = \Delta_1 y_4 - \Delta_1 y_3.$$

In view of the difference table, Eqs. (225) become

$$a_0 = y_n,$$

$$a_1 = \frac{\Delta_1 y_n}{h},$$

$$a_2 = \frac{\Delta_2 y_n}{2h^2},$$

etc.,

and Eq. (224) is

$$\begin{aligned} F(t) = y_n &+ \frac{\Delta_1 y_n}{h}(t - t_n) + \frac{\Delta_2 y_n}{2!\,h^2}(t - t_n)(t - t_{n-1}) \\ &+ \frac{\Delta_3 y_n}{3!\,h^3}(t - t_n)(t - t_{n-1})(t - t_{n-2}) + \dots \\ &+ \frac{\Delta_n y_n}{n!\,h^n}(t - t_n)(t - t_{n-1}) \dots (t - t_1). \end{aligned} \tag{226}$$

Last, approximate values of the integrals

$$\int_{t_{n-2}}^{t_{n-1}} f(t)dt, \quad \int_{t_{n-1}}^{t_n} f(t)dt, \quad \text{and} \quad \int_{t_n}^{t_{n+1}} f(t)dt$$

are required.

Formula (226) is called Newton's formula for backward interpolation. At the points (t_n, y_n), (t_{n-1}, y_{n-1}), . . . (t_1, y_1), Eq. (226) gives the values of the function $y = f(t)$ exactly. At intermediate points of the intervals $(t_n - t_{n-1})$, $(t_{n-1} - t_{n-2})$, and $(t_{n-2} - t_{n-3})$ the formula is used for interpolating values of $f(t)$. Moreover, it is used for extrapolating values to $y = f(t)$ in the interval $t_{n+1} - t_n$ provided this interval is sufficiently small.

Consequently, to find the approximate value of the last three integrals, replace $f(t)$ by $F(t)$. The resulting integrations are easily carried out by the substitution $t = t_n - hv$. Formula (226) then becomes

$$F(t) = y_n - \Delta_1 y_n v + \frac{\Delta_2 y_n}{2!} v(v - 1) - \frac{\Delta_3 y_n}{3!} v(v - 1)(v - 2) + \dots .$$

Then

$$\begin{aligned} \int_{t_{n-2}}^{t_{n-1}} f(t)dt &\doteq -h \int_2^1 \left[y_n - \Delta_1 y_n v + \frac{\Delta_2 y_n}{2!} v(v - 1) \right. \\ &\qquad \left. \frac{\Delta_3 y_n}{3!} v(v - 1)(v - 2) + \dots \right] dv \\ &\doteq h(y_n - \tfrac{3}{2}\Delta_1 y_n + \tfrac{5}{12}\Delta_2 y_n + \tfrac{1}{24}\Delta_3 y_n + \dots). \end{aligned} \tag{227}$$

Likewise,

$$\int_{t_{n-1}}^{t_n} f(t)dt \doteq h(y_n - \tfrac{1}{2}\,\Delta_1 y_n - \tfrac{1}{12}\,\Delta_2 y_n - \tfrac{1}{24}\,\Delta_3 y_n - \ldots), \quad (228)$$

$$\int_{t_n}^{t_{n+1}} f(t)dt \doteq h(y_n + \tfrac{1}{2}\,\Delta_1 y_n + \tfrac{5}{12}\,\Delta_2 y_n + \tfrac{3}{8}\,\Delta_3 y_n + \ldots). \quad (229)$$

64. Reduction of Systems of Equations to the Normal Form. The general theory of numerical integration is applicable only to differential equations in what is known as the normal form. This form consists of a system of simultaneous equations, the left members containing a single first derivative, while the right members contain no derivative. The number of equations in the normal form of the system equals the order of the system. (See § 18.) The reduction to the normal form is merely a routine process. One new dependent variable must be introduced for each differentiation of order higher than the first which occurs. The process is illustrated as follows:

EXAMPLE 1. The differential equations of the motion of a projectile, under proper assumptions, are

$$\frac{d^2x}{dt^2} = -k^2\frac{dx}{dt},$$

$$\frac{d^2y}{dt^2} = -k^2\frac{dy}{dt} - g,$$

where x and y are the coordinates of the projectile, t is the time, and

$$k^2 = \frac{H(y)G(V)}{C}.$$

The constant C is the ballistic coefficient dependent upon the shape of the projectile, $H(y)$ is a function of the height of the shell above ground, and $G(V)$ is a function of the velocity.

Reduce these equations to the normal form. Let

$$x = x_1, \qquad y = x_3,$$

$$\frac{dx}{dt} = x_2, \qquad \frac{dy}{dt} = x_4.$$

Then

$$\frac{dx_1}{dt} = x_2,$$

$$\frac{dx_2}{dt} = -k^2x_2,$$

$$\frac{dx_3}{dt} = x_4,$$

$$\frac{dx_4}{dt} = -k^2x_4 - g.$$

The last four equations are the normal form of the two second-order differential equations of motion of the projectile.

65. General Method of Numerical Integration of Differential Equations. The notation for the general theory for n normal equations is complicated. Accordingly, the exposition is given for the system

$$\begin{aligned} \frac{dx}{dt} &= f(x,y;t), \\ \frac{dy}{dt} &= g(x,y;t), \\ x = x_0, \quad y &= y_0, \quad t = 0. \end{aligned} \tag{230}$$

The method is easily extended to any number of equations. The exposition does not contain any proof that the processes employed actually converge to the solution required. Such a proof exists, but we are interested only in the steps required to obtain a solution. The carrying out of a numerical solution consists of two parts: (A) starting the solution, (B) continuing the solution. Each of these parts consists of several steps.

(A) *Starting the solution.*

1. The first step of the solution consists in choosing an increment h of t, which will be the time-interval between successive desired points of the solution. No definite rule for determining a good size for h can be given, but a method for ascertaining whether any chosen value is too great or too small will appear later.

2. In the next step, we calculate $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at $t = 0$ from (230).

3. We now assume that, over the small interval h, the first derivatives are approximately constant, and obtain

$$\begin{aligned} x_1^{(1)} &= x_0 + hf_0, \\ y_1^{(1)} &= y_0 + hg_0. \end{aligned} \tag{231}$$

The superscript (1) indicates that this is the first approximation, and $f_0 = f(x_0, y_0; 0)$, $g_0 = g(x_0, y_0; 0)$.

4. From the approximate values of x_1 and y_1 given by (231), we calculate $f_1^{(1)}$ and $g_1^{(1)}$ from (230).

5. A better approximation to x_1 and y_1 may now be obtained by assuming that the average slope from point 0 to point 1 is the average of the slopes at these points. This gives

$$\begin{aligned} x_1^{(2)} &= x_0 + \tfrac{1}{2}\, h(f_0 + f_1^{(1)}), \\ y_1^{(2)} &= y_0 + \tfrac{1}{2}\, h(g_0 + g_1^{(1)}). \end{aligned} \tag{232}$$

6. Using the second approximations given by (232), new approximations to f_1 and g_1 are calculated.

7. From the new approximations $f_1^{(2)}$ and $g_1^{(2)}$, third approximations of x_1 and y_1 are calculated:

$$\begin{aligned} x_1^{(3)} &= x_0 + \tfrac{1}{2}\, h(f_0 + f_1^{(2)}), \\ y_1^{(3)} &= y_0 + \tfrac{1}{2}\, h(g_0 + g_1^{(2)}). \end{aligned} \tag{233}$$

8. From $x_1^{(3)}$ and $y_1^{(3)}$ the values of $f_1^{(3)}$ and $g_1^{(3)}$ are calculated from (230).

9. Steps 3 to 8 are now repeated for the second interval, the results being $x_2^{(3)}$, $y_2^{(3)}$, $f_2^{(3)}$, $g_2^{(3)}$. At this stage (x_0, y_0), (x_1, y_1), and (x_2, y_2) are all known; the first pair exactly and the others approximately. We now correct these approximations.

10. We construct difference tables for f and g:

$$\begin{array}{lllllll} t_0 & f_0 & & & g_0 & & \\ t_1 & f_1 & \Delta_1 f_1 & & g_1 & \Delta_1 g_1 & \\ t_2 & f_2 & \Delta_1 f_2 & \Delta_2 f_2 & g_2 & \Delta_1 g_2 & \Delta_2 g_2 \end{array} \tag{234}$$

11. Now the exact values of x_1 and y_1 are

$$x_1 = x_0 + \int_{t_0}^{t_1} f dt, \qquad y_1 = y_0 + \int_{t_0}^{t_1} g dt. \tag{235}$$

(This is seen to be true, since, if

$$x = x_0 + \int_{t_0}^{t} f dt, \text{ and } y = y_0 + \int_{t_0}^{t} f dt$$

are substituted in (230), the differential equations are satisfied. The values of x and y given by the last two equations are for any time t.

When $t = t_1$, then $x = x_1$ and $y = y_1$.) We apply (227) to find corrected values of x and y. In (227), let $n = 2, y = f$, and obtain

$$\int_{t_0}^{t_1} f dt = h(f_2 - \tfrac{3}{2}\,\Delta_1 f_2 + \tfrac{5}{12}\,\Delta_2 f_2) \tag{236}$$

and a similar result for the integral of g.

12. By substituting (236) into (235), we obtain $x_1^{(4)}$ and $y_1^{(4)}$, from which and (230) we calculate $f_1^{(4)}$ and $g_1^{(4)}$.

13. Next, repeat steps 9, 10, and 11, if necessary.

14. Since the exact values of x_2 and y_2 are

$$x_2 = x_1 + \int_{t_1}^{t_2} f dt, \quad y_2 = y_1 + \int_{t_1}^{t_2} g dt, \tag{237}$$

we apply (228) with $n = 2$, obtaining

$$\int_{t_1}^{t_2} f dt = h(f_2 - \tfrac{1}{2}\,\Delta_1 f_2 - \tfrac{1}{12}\,\Delta_2 f_2) \tag{238}$$

and a similar result for g.

15. By substituting (238) into (237), we obtain new approximations to x_2 and y_2, from which we calculate new values of f_2 and g_2.

16. The new values of f_2 and g_2 and their differences may be used to recalculate x_1 and y_1; but if the value chosen for h is as small as it should be this will be unnecessary, and it may be assumed that the values at points 0,1,2 are correct and that the solution is fully started.

(*B*) *Continuing the solution.*

17. The new values of f and g are used to construct a new difference table similar to (234).

18. By an application of (229) with $n = 2$, values of x_3 and y_3 are extrapolated:

$$x_3^{(1)} = x_2 + \int_{t_2}^{t_3} f dt = x_2 + h(f_2 + \tfrac{1}{2}\,\Delta_1 f_2 + \tfrac{5}{12}\,\Delta_2 f_2) \ldots . \tag{239}$$

19. From the values $x_3^{(1)}$ and $y_3^{(1)}$, calculate values of $f_3^{(1)}$ and $g_3^{(1)}$ from (230), and append these and their differences on to the table of step 17. Third differences now enter.

20. Apply (228) with $n = 3$ to correct x_3 and y_3, obtaining

$$x_3' = x_2 + \int_{t_2}^{t_3} f dt = x_2 + h(f_3 - \tfrac{1}{2}\,\Delta_1 f_3 - \tfrac{1}{12}\,\Delta_2 f_3 - \tfrac{1}{24}\,\Delta_3 f_3) \ldots . \tag{240}$$

21. Repeat steps 19 and 20 if necessary.

22. The above steps are now repeated for the next interval, and the solution is well under way.

From the preceding steps it will be seen that Eq. (227) is used to check the next to the last point, (228) to check the last point, and (229) to extrapolate to the next point.

Tables of differences for x and y are unnecessary, but are useful in checking the work. If these differences become irregular it is probable that an error has been made.

If the third differences of f and g are approximately constant the interval h may be doubled. On the other hand, if they are quite irregular h should be halved. This may be done by interpolating points half-way between the calculated points by means of (226).

If several trial calculations (step 21 or steps 7 and 8 above) are necessary before an accurate value is obtained, the value of h should be halved as explained above.

Example on the general method. In order that the general method may be compared with the methods of §§ 61–62, we again solve the example of § 61. Eqs. (230) for this example are

$$\frac{dx}{dt} = 0.9 - \sin y = f(x,y;t),$$
$$\frac{dy}{dt} = x = g(x,y;t), \tag{241}$$

$$y_0 = 53° = 0.925 \text{ radian}, \; x_0 = 0, \; t_0 = 0.$$

The numbers of the statements below refer to the corresponding numbers of the steps of § 65. *Many steps may be omitted unless excessive accuracy is desired.*

1. h 1 second.
2. f_0 0.1014, $g_0 = 0$.
3. $x_1^{(1)} = 0.1014$
   $y_1^{(1)} = 0.925$
4. $f_1^{(1)} = 0.1014$
   $g_1^{(1)} = 0.1014$
5. $x_1^{(2)} = 0.1014$
   $y_1^{(2)} = 0.9757$
6. $f_1^{(2)}$ 0.0719
   $g_1^{(2)} = 0.1014$
7. $x_1^{(3)} = 0.0866$
   $y_1^{(3)} = 0.9757$
8. $f_1^{(3)} = 0.0719$
   $g_1^{(3)} = 0.0866$
9. $x_2^{(1)} = 0.1585$
   $y_2^{(1)} = 1.062$
   $f_2^{(1)} = 0.0267$
   $g_2^{(1)} = 0.1585$
   $x_2^{(2)} = 0.1359$
   $y_2^{(2)} = 1.0983$
   $f_2^{(2)} = 0.0096$
   $g_2^{(2)} = 0.1359$
   $x_2^{(3)} = 0.1274$
   $y_2^{(3)}$ 1.0869
   $f_2^{(3)}$ 0.0148
   $g_2^{(3)} = 0.1274$

10. Difference table for f

t	f	$\Delta_1 f$	$\Delta_2 f$
t_0	0.1014		
t_1	0.0719	−0.0295	
t_2	0.0148	−0.0571	−0.0276

11. Difference table for g

t	g	$\Delta_1 g$	$\Delta_2 g$
t_0	0		
t_1	0.0866	0.0866	
t_2	0.1274	0.0408	−0.0458

12. $$\begin{aligned} x_1^{(4)} &= 0 + [0.0148 + \tfrac{3}{2}\,(0.0571) - \tfrac{5}{12}\,(0.0276)] \\ &= 0.0889 \\ y_1^{(4)} &= 0.9250 + [0.1274 - \tfrac{3}{2}\,(0.0408) - \tfrac{5}{12}\,(0.0458)] \\ &= 0.9721 \end{aligned}$$

14. $$\begin{aligned} x_2^{(4)} &= 0.0889 + [0.0148 + \tfrac{1}{2}\,(0.0571) + \tfrac{1}{12}\,(0.0276)] \\ &= 0.1346 \\ y_2^{(4)} &= 0.9721 + [0.1274 - \tfrac{1}{2}\,(0.0408) + \tfrac{1}{12}\,(0.0459)] \\ &= 1.0829 \end{aligned}$$

It is unnecessary to recalculate f_2 and g_2.

18. $$\begin{aligned} x_3^{(1)} &= 0.1346 + [0.0148 - \tfrac{1}{2}\,(0.0571) - \tfrac{5}{12}\,(0.0276)] \\ &= 0.1094 \\ y_3^{(1)} &= 1.0829 + [0.1274 + \tfrac{1}{2}\,(0.0408) - \tfrac{5}{12}\,(0.0458)] \\ &= 1.2117 \end{aligned}$$

19. $$\begin{aligned} f_3^{(1)} &= -\,0.0362 \\ g_3^{(1)} &= 0.1094 \end{aligned}$$

t	f	$\Delta_1 f$	$\Delta_2 f$	$\Delta_3 f$
t_0	0.1014			
t_1	0.0719	−0.0295		
t_2	0.0148	−0.0571	−0.0276	
t_3	−0.0362	−0.0510	0.0061	0.0337

t	g	$\Delta_1 g$	$\Delta_2 g$	$\Delta_3 g$
t_0	0			
t_1	0.0866	0.0866		
t_2	0.1274	0.0408	−0.0458	
t_3	0.1094	−0.0180	−0.0588	−0.013

20.

$$x_3^{(2)} = 0.1346 - 0.0362 + \tfrac{1}{2}(0.0510) - \tfrac{1}{12}(0.0061) - \tfrac{1}{24}(0.0337)$$
$$= 0.1220$$
$$y_3^{(2)} = 1.0829 + 0.1094 + \tfrac{1}{2}(0.0180) + \tfrac{1}{12}(0.0588) + \tfrac{1}{24}(0.0130)$$
$$= 1.2067$$

The solution is now started, and additional points are simply found. If the four points obtained are plotted in the graph of Fig. 32, they are seen to give very approximately the results previously obtained.

66. Summary. The methods of §§ 60–63 should be tried first. If the system of equations is too complicated, the use of the general method of § 65 may be necessary. If the problem is one based on electrical phenomenon and if the dependent variables are currents or voltages, the results obtained by numerical integration contain only the information available from an oscillogram. If the problem is a mechanical one and if the system of differential equations contains many parameters (say 12), this requiring several integrations, recourse may be had to the differential analyzer.[15] The results in this case are a book of curves; the parameters have been lost from the solution. However, the engineer is interested in the behavior of systems for various values of parameters. Formulas are needed. Consequently analytical methods are essential. The differential equations of problems 4 and 5 below, along with many others, are solvable analytically. (See Ref. 6). Appreciation of analytical methods is enhanced by one or two attempts at numerical integration where the general method of § 65 must be employed.

PROBLEMS

1. The rate of build-up of the armature voltage of a separately excited d-c. generator with field coils in parallel is given by the relation

$$\frac{dE_A}{dt} = \frac{(E - i_F R_F)}{K \times N_F},$$

where E_A = armature voltage,
E = a constant voltage suddenly applied to the field,
i_F = field current per winding,
R_F = field resistance per winding,
N_F = number of field turns per winding,
K = a constant depending on design.

If the saturation curve is available and the machine constants are known, the value of dE_A/dt may be obtained for any value of E_A by substituting the value of i_F corresponding to E_A (from the saturation curve).

[15] See *Journal of the Franklin Institute*, October, 1931.

The reciprocal of dE_A/dt, namely, dt/dE_A, may then be plotted as a function of E_A. The area under this curve is evidently t; i.e., the time required for the armature voltage to build up to any value is the area under the curve up to that value of armature voltage. This area may be found by numerical integration.

Given that $E = 125$ volts (applied at t 0),
$R_F = 6$ ohms,
$N_F = 750$ turns,
$K = 14{,}140 \times 10^{-8}$,

and that the saturation curve is determined by the following data:

Armature Voltage	Field Current per Pole	Armature Voltage	Field Current per Pole
65	1	305	7
130	2	316	8
189	3	325	9
235	4	333	10
267	5	341	11
289	6		

plot the curve of armature voltage as a function of time.

2. Solve the equation $\dfrac{d^2y}{dt^2} = 0.9 - \sin y$ by the method of § 62 and without making any substitution for change of scale.

3. A sphere moving with a small velocity in a fluid experiences a resisting force $F = 6\pi\rho rv$. For any velocity the resistance is $F = \frac{1}{2}\pi\rho r^2v^2C_w(R)$, where r is the radius of the sphere, v its velocity, μ the viscosity of the fluid, ρ its density, $R = \rho rv/\mu$, and $C_w(R)$ is a function of R given below. The differential equation describing the motion due to gravity in a fluid is

$$\frac{dv}{dt} = g - \frac{F}{M},$$

where g is the acceleration of gravity and M the mass of the sphere.

An iron sphere 20 cm. in diameter is released from a height of 1000 meters. How much longer will it take to reach the ground than if it were falling in vacuum? The following data are in c.g.s. units:

R	$C_w(R)$	R	$C_w(R)$
10^{-1}	58	10^4	1.1
1	10	10^5	1.2
10	2.7	2×10^5	1.15
10^2	1.4	5×10^5	0.3
10^3	1.0	10^6	0.34

The fluid is water at 4 °C.

4. Solve by graphical or numerical integration the following system of differential equations.

$$\frac{dI}{dt} = \frac{(RI - E)\left[\left(\frac{rs_0}{s}\right)^2 + x_d x_q\right]}{L\left[\left(\frac{rs_0}{s}\right)^2 + x'_d x_q\right]}$$

$$- \frac{2KPr^3I^3(x_d - x'_d)\left[x_q^2 + \left(\frac{rs_0}{s}\right)^2\right]x_q s_0^2}{JI_0^2\left[\left(\frac{rs_0}{s}\right)^2 + x_d x_q\right]^3 s^4\left[\left(\frac{rs_0}{s}\right)^2 + x'_d x_q\right]},$$

$$\frac{ds}{dt} = \frac{-KPrI^2}{JsI_0^2}\left[\frac{x_q^2 + \left(\frac{rs_0}{s}\right)^2}{\left[\left(\frac{rs_0}{s}\right)^2 + x_d x_q\right]^2}\right].$$

The initial conditions are $I = I_0$, $s = s_0$ at $t = 0$. These are the equations of dynamic braking of synchronous machines. The armature is short-circuited through a resistance r. The dependent variables are the speed s and field current I. The constants are:

$x_d = 0.64$.	I_1 = jump of current = 32.
$x_q = 0.46$.	$r = 0.227$.
$x'_d = 0.29$.	P = power = 400 *kv-a*.
$\frac{E}{R}$ 85.	$K = 735.5$.
	$L = 0.67$ henry.
$R = 0.002$ ohm.	J = moment of inertia 232,000.
I_0 = no load current 57.5.	$s_0 = 94.8$ r.p.m.

First make the following change of dependent variables:

$$s = s_0 e^{-z},$$

$$I = \frac{E}{R} - I_1 e^{-y}.$$

5. Solve by numerical integration the equation

$$\left(\frac{NA}{K(A + Bi)^2\, 10^8}\right)\frac{di}{dt} + ri = \frac{i(a - be^{-at})}{A + Bi}.$$

This is an equation for the field current i of a shunt excited d-c. generator which undergoes an exponential change of speed. The symbols have the following numerical values:

N number of field turns	= 4500.
K voltage proportionality factor	= 7×10^{-8} volt per line per rev. per min.
r field circuit resistance	= 55 ohms.

a final speed	$= 2400$ r.p.m.
b total change of speed	$= 1200$ r.p.m.
α reciprocal time constant of speed change	$= 1$ sec.$^{-1}$
A constant	$= 15.$
B constant	$= 1.5.$

A and B are constants in the equation used to approximate the magnetization curve:

$$\phi = \frac{i}{K(A + Bi)}.$$

CHAPTER III

VECTOR ANALYSIS

The chief uses of vector analysis in engineering are: derivation of the partial differential equations of mathematical physics; the study of vector fields (magnetic, electrostatic, and hydrodynamic); and the analysis of rotating electrical machines. Since the emphasis of this text is on the reduction of physical phenomena to mathematical form, we are interested only in those parts of vector analysis which assist in these reductions.

I

OPERATORS AND LAWS OF VECTOR ANALYSIS

The application of vector analysis to engineering problems can readily be made only after certain notations and laws of manipulation are understood. The first section of this chapter is thus necessarily concerned with a brief introduction to the purely formal parts of vector analysis. Some proofs are left as exercises which appear at the end of the section.

67. Vectors. A **vector** is a quantity that possesses direction as well as magnitude; a **scalar** is a quantity that possesses magnitude only. Quantities such as mass, temperature, and electric charge are scalars; velocity, force, and current density are vectors. Vector analysis deals with vectors which are defined at a single point as well as with the more general case of vectors defined at more than one point, as along a line, on a plane, or in a volume. When the vectors are defined at a single point, as in the treatment of forces on a rigid body, the laws of vector algebra, addition, subtraction, and multiplication, are applied. For example, the sum of vector forces and reactions is zero; and if all but one are known, the remaining quantities can be determined. Another example of the use of vectors defined at one point is the vector treatment of alternating currents. Here the length of the vector is proportional to the amplitude of the current or voltage and the angular displacement to the relative phase. The vectors are usually confined to one plane and admit of the usual operations

of vector algebra. The methods of complex number theory are also used to treat the subject of alternating currents.

In more general vector problems, in which a vector is defined for all points in a given region, the principles of vector calculus may be applied, as well as the algebra of vectors. Such a region is a **vector field**, for the discussion of which certain theorems are available.

The subject of vector fields includes gravitational force fields, electric and magnetic flux densities, magnetic vector potential, the Poynting vector, current density in a solid conductor, temperature gradient, and others. The electric and magnetic fields because of their importance are chosen for discussion, and other fields are treated by analogy.

68. Nature of Vector Analysis. In §§ 10–20, we have seen how the first and second derivatives of the calculus can be combined, by means of Newton's laws of motion and Kirchhoff's laws, to describe engineering phenomena. Our interest in vector analysis is much the same. There exist in vector analysis a number of operators called the gradient, divergence, curl, line and surface integrals, etc., which play the same rôle in vector analysis that derivatives and integrals play in differential equations. These new operators possess physical significance just as do the derivatives. By giving the values of certain of these operators throughout a region, a vector field is completely described, just as a function in calculus is completely determined (except for a constant) for real values of the independent variable, by giving its derivative.

The laws of physics and engineering make possible the combination of these operators in equations in much the same way that Newton's and Kirchhoff's laws combine derivatives in equations. In the calculus, the derived equations are ordinary differential equations. In vector analysis, the derived equations are partial differential equations. After a partial differential equation is obtained, vector analysis is, in general, of no further use. The solution of the equation belongs to another branch of mathematics. The derivation of the equation requires not only a knowledge of vector analysis, but also some knowledge of physics and engineering. However, it is usually far easier to derive a partial differential equation than to solve it.

69. Algebra of Vectors. Vector algebra is very similar to scalar algebra.

(*a*) *Definitions.* **Zero** and **unit vectors** are those whose magnitudes are respectively zero and one. Two vectors are equal, if and only if, they have the same magnitude and direction. By the negative vector $-\mathbf{A}$, we mean $\mathbf{A}$ with its direction reversed but its magnitude

unchanged. A vector **A** may always be considered as $A\mathbf{a}$, where **a** is a unit vector and A is the magnitude of **A**.

(*b*) *Addition and subtraction.* **C**, the sum of **A** and **B**, is defined to be the vector obtained by placing the initial point of **B** in coincidence with the terminal of **A** and taking **C** with its initial point coinciding with that of **A**, and its terminal point with that of **B**. From Fig. 34, evidently $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$. The sum of three vectors $\mathbf{E} = \mathbf{A} + \mathbf{B} + \mathbf{D} = \mathbf{C} + \mathbf{D}$, where $\mathbf{A} + \mathbf{B} = \mathbf{C}$. The subtraction of **A** is defined to be the addition of $-\mathbf{A}$.

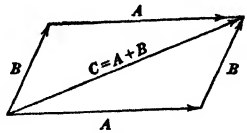


FIG. 34.

(*c*) *Vector components.* A vector is uniquely determined by giving its projections on the three coordinate axes. These projections are $A_x = A \cos (Ax)$, $A_y = A \cos (Ay)$, $A_z = A \cos (Az)$, where (Ax) denotes the angle between **A** and the positive x-axis. If $\mathbf{A} + \mathbf{B} = \mathbf{C}$, it is apparent geometrically that

$$A_x + B_x = C_x,$$
$$A_y + B_y = C_y,$$
$$A_z + B_z = C_z.$$

Let, **i**, **j**, and **k** be unit vectors coinciding with the x, y, and z axes respectively. By the definition of addition

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}.$$

(*d*) *Scalar and vector products.* The scalar product of **A** by **B** (or **B** by **A**) is a scalar defined by the equation $\mathbf{A}\cdot\mathbf{B} = AB \cos \theta$, where θ is the angle between the positive directions of **A** and **B**. The scalar product is thus the product of one vector by the projection of the other vector upon it. Hence $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$. Also

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1, \quad \text{and} \quad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0.$$

It can be shown that $\mathbf{A}\cdot(\mathbf{B} + \mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$; thus we may write

$$\begin{aligned} A\cdot B &= (\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z)\cdot(\mathbf{i}B_x + \mathbf{j}B_y + \mathbf{k}B_z) \\ &= \mathbf{i}\cdot\mathbf{i}A_xB_x + \mathbf{i}\cdot\mathbf{j}A_xB_y + \mathbf{i}\cdot\mathbf{k}A_xB_z \\ &\quad + \mathbf{j}\cdot\mathbf{i}A_yB_x + \mathbf{j}\cdot\mathbf{j}A_yB_y + \mathbf{j}\cdot\mathbf{k}A_yB_z \\ &\quad + \mathbf{k}\cdot\mathbf{i}A_zB_x + \mathbf{k}\cdot\mathbf{j}A_zB_y + \mathbf{k}\cdot\mathbf{k}A_zB_z \\ &= A_xB_x + A_yB_y + A_zB_z. \end{aligned}$$

The vector product of **A** by **B** (not **B** by **A**) is a vector defined by the equation

$$\mathbf{A} \times \mathbf{B} = \mathcal{E}AB \sin \theta,$$

where θ is the angle between the positive directions of **A** and **B** and $\mathcal{E}$ is a unit vector perpendicular to the plane of **A** and **B**. The positive direction of **A** $\times$ **B** *is defined to be perpendicular to the plane of* **A** *and* **B** in the sense of advance of a right-handed screw rotated from the first to the second of these vectors through the smaller angle between their positive directions. (See Fig. 35.)

$$\mathbf{A} \times \mathbf{B} = \mathcal{E} AB \sin \theta.$$

Consequently, $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$, and $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \boldsymbol{i}$, and $\boldsymbol{k} \times \mathbf{i} = \mathbf{j}$. Also, $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$. It is evident that the vector product of **A** and **B** may be considered as a vector with a magnitude equal to the area of the parallelogram having **A** and **B** as sides and with the direction of the normal to the plane of **A** and **B**.

It can be proved that the distributive law of multiplication, namely $(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \times \mathbf{C}) + (\mathbf{B} \times \mathbf{C})$, holds for vector products (as well

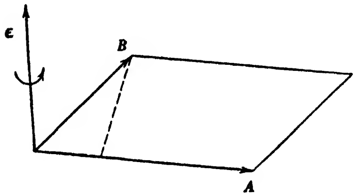


FIG. 35.

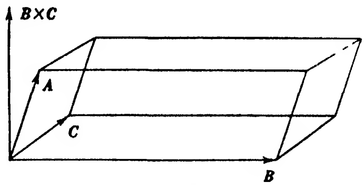


FIG. 36.

as for scalar products). In view of this and the above relations between **i**, **j**, and **k**, we may express **A** $\times$ **B** in terms of its **i**, **j**, **k** components as follows:

$$\begin{aligned}
\mathbf{A} \times \mathbf{B} &= (\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z) \times (\mathbf{i}B_x + \mathbf{j}B_y + \mathbf{k}B_z) \\
&= \mathbf{i} \times \mathbf{i}A_xB_x + \mathbf{i} \times \mathbf{j}A_xB_y + \mathbf{i} \times \mathbf{k}A_xB_z \\
&\quad + \mathbf{j} \times \mathbf{i}A_yB_x + \mathbf{j} \times \mathbf{j}A_yB_y + \mathbf{j} \times \mathbf{k}A_yB_z \\
&\quad + \mathbf{k} \times \mathbf{i}A_zB_x + \mathbf{k} \times \mathbf{j}A_zB_y + \mathbf{k} \times \mathbf{k}A_zB_z \\
&= \mathbf{i}(A_yB_z - A_zB_y) + \mathbf{j}(A_zB_x - A_xB_z) \\
&\qquad + \mathbf{k}(A_xB_y - A_yB_x).
\end{aligned}$$

The vector product may be written as the determinant

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

(*e*) *Triple scalar product.* The product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is a scalar called the triple scalar product. Inspection of Fig. 36 shows that it is equal to the volume of a parallelopiped with edges **A**, **B**, and **C**.

Since interchanging the terms in a scalar product does not change the sign of the product whereas interchanging the terms in a vector product does change the sign of the product, it follows that

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{B}\times\mathbf{C})\cdot\mathbf{A} = -(\mathbf{C}\times\mathbf{B})\cdot\mathbf{A} = -\mathbf{A}\cdot(\mathbf{C}\times\mathbf{B}).$$

Since the volume of the parallelopiped remains the same, no matter which face is considered as base, it follows that

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}), \text{ etc.}$$

Thus the dot and cross may be interchanged at will and the sign of the product remains unchanged so long as the cyclic order of the vectors remains the same. The triple scalar product may be written as

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{B}\times\mathbf{C})\cdot\mathbf{A} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

(*f*) *Triple vector product.* The product $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is defined as the triple vector product. The vector product of $\mathbf{B}\times\mathbf{C}$ should be formed first, and then the vector product of **A** with this result. The final result may be shown to be

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}). \quad \text{(See Ex. 3, § 76.)}$$

70. Line and Surface Integrals Involving Vectors. Certain definitions of curl and divergence are based upon the ideas of line and surface integrals involving vectors.

(*a*) *Line integrals.* The integral $\int_A F\cdot d\mathbf{r}$ is a line integral. The vector $d\mathbf{r}$ is taken along the tangent to the curve AB as in Fig. 37, and the vector **F** may vary in both magnitude and direction along the curve. Alternative forms are

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} = \int_A^B F\cos\theta\, dr,$$

and

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} = \int_A^B (\mathbf{i}F_x + \mathbf{j}F_y + \mathbf{k}F_z)\cdot(\mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz)$$

$$= \int_A^B (F_x dx + F_y dy + F_z dz).$$

If $\mathbf{F}$ represents a force on a body, $\int_A^B \mathbf{F}\cdot d\mathbf{r}$ is the work done by the force as the body moves over the specified path from A to B.

EXAMPLE 1. To fix the ideas more clearly, let $\mathbf{F}$ be the force of gravity. Let the curve AB (Fig. 38) be one-quarter of the circumference of a circle. Let us determine the work W done in moving a mass m against the force of gravity from A to B along the curve AB in the zy-plane. There is no friction. Then $\mathbf{F} = -mg\mathbf{k}$. (The minus

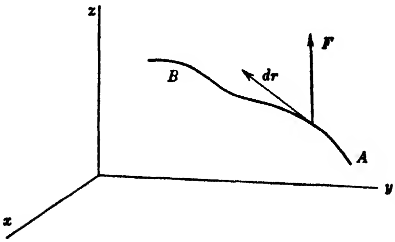


FIG. 37.

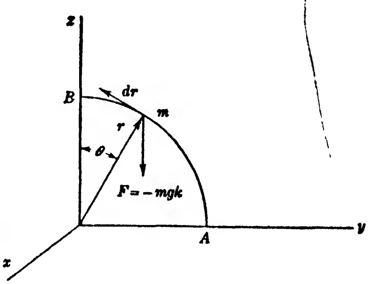


FIG. 38.

sign is due to the fact that the force is in the direction of negative $\mathbf{k}$) Then,

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z = \mathbf{j}y + \mathbf{k}z,$$

$$d\mathbf{r} = \mathbf{j}dy + \mathbf{k}dz,$$

$$z = r\cos\theta,$$

$$dz = -r\sin\theta d\theta,$$

and

$$W = -\int_A^B \mathbf{F}\cdot d\mathbf{r} = -\int_A^B (-mg\mathbf{k})\cdot(\mathbf{j}dy + \mathbf{k}dz)$$

$$= \int_0^{\pi/2} (-mg\mathbf{k})\cdot(-r\sin\theta d\theta)\mathbf{k} = mgr\int_0^{\pi/2} \sin\theta d\theta = mgr.$$

This is, of course, the work done in raising the mass M a vertical distance r. If $\mathbf{F}$ varied both in magnitude and direction and C were a complicated curve, the integrations would, in general, be more complicated, but no additional principles would be involved.

(b) *Surface integrals*. The integral $\int_S\int \mathbf{F}\cdot\mathbf{n}dS = \int_S\int \mathbf{F}\cdot d\mathbf{S}$ is the surface integral of $\mathbf{F}$ over the surface S. Let the curved surface

of Fig. 39 be divided into infinitesimal rectangles ΔS. The elements so formed may be treated as if plane. A plane surface may be represented by a vector whose magnitude is equal to the area of the surface and whose direction is that of the positive normal.

If the elementary plane surface is part of a closed surface, the positive direction of the normal is outward. Denote by $\mathbf{n}$ the unit normal to the surface. Then the vector representing the elementary surface in Fig. 39 is $d\mathbf{S} = \mathbf{n}\Delta S$.

The integral $\int_S\int \mathbf{F}\cdot\mathbf{n}dS$ has many applications as well as being fundamental in definitions. It represents volume rate of flow through the surface S if $\mathbf{F}$ is a velocity vector. Consider an incompressible liquid flowing through the surface of Fig. 39. At the point $P(x, y, z)$ let its velocity be $\mathbf{F}$, parallel to the z-axis as shown. Since the liquid is incompressible, the flow through the element of surface ΔS per unit time will be the same as the flow through the element ΔS_z (see Fig. 39) per unit time. But $\Delta S_z = \Delta S \cos\theta$, where θ is the angle between $\mathbf{n}$ and the z-axis. Hence, the rate of flow is

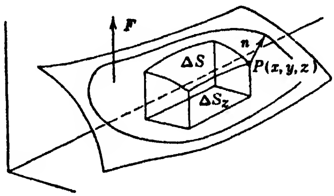


FIG. 39.

$$\Delta S_z F = \Delta S(\cos\theta)F$$
$$= \Delta S\mathbf{F}\cdot\mathbf{n},$$

or the total flow per unit time through the whole surface is

$$\text{Flow} = \int\int \mathbf{F}\cdot\mathbf{n}dS.$$

The surface integral of $\mathbf{F}$ over a surface is called the **flux** of $\mathbf{F}$ through the surface. For example, if heat is flowing through a surface S, the amount of heat which crosses unit area drawn normally to the lines of flow in unit time is called the **intensity of heat flow or the heat-current density, q,** while $\int\int \mathbf{q}\cdot d\mathbf{S}$ is the **flux of heat.**

EXAMPLE 2. Let the velocity $\mathbf{F}$ in Fig. 39 be defined by the equation $\mathbf{F} = 3y\mathbf{k}$. Let the surface considered be the octant of a sphere, whose equation is $x^2 + y^2 + z^2 = R^2$. The flow through the surface per unit time then is

$$\text{Flow} = \int_S\int \mathbf{F}\cdot\mathbf{n}dS = \int\int 3y\mathbf{k}\cdot\frac{(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)}{R}dS = \int_S\int \frac{3yz}{R}dS.$$

The spherical surface coordinates R, θ, ϕ, are related to x, y, z by the relations:

$$x = R \cos \phi \cos \theta,$$

$$y = R \cos \phi \sin \theta,$$

$$z = R \sin \phi.$$

From Fig. 40, dS is seen to be $R^2 \cos \phi d\theta d\phi$. Hence

$$\text{Flow} = \int_0^{\pi/2} \int_0^{\pi/2} 3R^3 \cos^2 \phi \sin \theta \sin \phi d\theta d\phi = R^3.$$

Since, in the case of an incompressible fluid, the flow through the trace in the xy-plane is equal to the flow through the spherical surface, the above result is easily checked by the integral

$$\text{Flow} = \int_0^R 3yxdy = \int_0^R 3(R^2 - y^2)^{\frac{1}{2}} ydy = R^3.$$

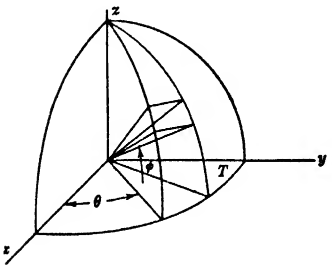


FIG. 40.

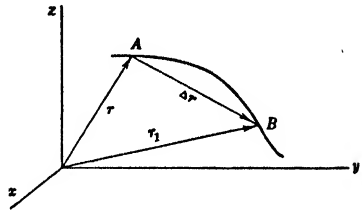


FIG. 41.

71. Vector Operators. Arts. 68–70 are the basis for the definitions and interpretations of derivative, gradient, divergence, and curl. These concepts, along with relations and theorems involving them, make up the nucleus of vector analysis. Many important concepts in mathematics have two or more equivalent definitions. That one is then employed which is most readily useful in a given situation. In §§ 73–75, two definitions of gradient and three each of divergence and curl appear. Arts. 72–78 may properly be called the calculus of vectors.

72. Derivatives of Vector Quantities. Let $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$, where $x = x(T)$, $y = y(T)$, $z = z\ (T)$, and T is any real parameter. If the initial point of $\mathbf{r}$ is fixed at the origin, the terminal point of $\mathbf{r}$ varies and describes a space curve as T varies. Let A and B be two nearby points on this curve. (See Fig. 41.) Then

$$\begin{aligned}\Delta\mathbf{r} = \overline{AB} &= \mathbf{r}_1 - \mathbf{r} \qquad (\mathbf{r}_1 \text{ is not a unit vector.})\\ &= \mathbf{i}x_1 + \mathbf{j}y_1 + \mathbf{k}z_1 - \mathbf{i}x - \mathbf{j}y - \mathbf{k}z\\ &= \mathbf{i}(x_1 - x) + \mathbf{j}(y_1 - y) + \mathbf{k}(z_1 - z)\\ &= \mathbf{i}\Delta x + \mathbf{j}\Delta y + \mathbf{k}\Delta z.\end{aligned}$$

Dividing ΔT and taking the limit

$$\frac{d\mathbf{r}}{dT} = \mathbf{i}\frac{dx}{dT} + \mathbf{j}\frac{dy}{dT} + \mathbf{k}\frac{dz}{dT}.$$

It is evident that, as B approaches A (Fig. 41), the vector representing $\Delta\mathbf{r}$ approaches the position of the tangent to the curve at A. Hence, $\dfrac{d\mathbf{r}}{dT}$ is a vector tangent to the space curve described by the terminus of $\mathbf{r}$. Thus, it follows that the derivative of a vector having constant magnitude but variable direction is a vector perpendicular to the differentiated vector.

By treatment similar to the above,

$$\frac{d^2\mathbf{r}}{dT^2} = \mathbf{i}\frac{d^2x}{dT^2} + \mathbf{j}\frac{d^2y}{dT^2} + \mathbf{k}\frac{d^2z}{dT^2}.$$

Formulas for differentiating $\mathbf{P}\cdot\mathbf{Q}$ and $\mathbf{P}\times\mathbf{Q}$ may be obtained by expressing each product in its expanded form and taking derivatives of these forms. Thus,

$$\begin{aligned}\frac{d}{dT}(\mathbf{P}\cdot\mathbf{Q}) &= \frac{dP_x}{dT}Q_x + \frac{dP_y}{dT}Q_y + \frac{dP_z}{dT}Q_z + P_x\frac{dQ_x}{dT} + P_y\frac{dQ_y}{dT} + P_z\frac{dQ_z}{dT}\\ &= \frac{d\mathbf{P}}{dT}\cdot\mathbf{Q} + \mathbf{P}\cdot\frac{d\mathbf{Q}}{dT},\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dT}(\mathbf{P}\times\mathbf{Q}) &= \mathbf{i}\left[\frac{dP_y}{dT}Q_z - \frac{dP_z}{dT}Q_y + P_y\frac{dQ_z}{dT} - P_z\frac{dQ_y}{dT}\right] + \text{etc.}\\ &= \frac{d\mathbf{P}}{dT}\times\mathbf{Q} + \mathbf{P}\times\frac{d\mathbf{Q}}{dT}.\end{aligned}$$

Both products are differentiated by differentiating the factors just as in the case of algebraic products, paying no attention to the dot or cross. It is important to notice, however, that in taking the derivative

of the vector product the order of the vectors must not be changed unless the sign is changed.

73. Gradient. Let $V(x, y, z)$ be a scalar point function. Suppose, for example, V is the temperature at any interior point $P(x, y, z)$ of a body. In general the scalar V will have different values at neighboring points; and the rate of change of V with respect to distance will depend upon the direction in which the distance is measured. Two useful and equivalent definitions for the gradient of a scalar are the following:

1. The gradient of V at point P is the vector having the direction of the greatest rate of increase of V, with respect to distance, at P, and a magnitude equal to this rate of increase. The symbol for the gradient of V is ∇V.

2. The gradient of V is also defined with respect to cartesian coordinates

$$\nabla V \equiv \mathbf{i}\frac{\partial V}{\partial x} + \mathbf{j}\frac{\partial V}{\partial y} + \mathbf{k}\frac{\partial V}{\partial z}.$$

By this definition the symbol ∇, called " del," is equivalent to a vector operator,

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

Two definitions are said to be equivalent if each implies the other. The equivalence of these two definitions is shown as follows. Consider the neighboring surfaces $V(x, y, z) =$ constant C passing through P, and $V(x, y, z) = C + dC$. (Fig. 42.) The normal derivative of V, written dV/dn, is the rate of change of V along the normal to the surface. The rate of change of V in any other direction $d\mathbf{r}$, dV/dr, is less than dV/dn because $dV = dC$ is constant on passing from C to $C + dC$ and $dn < dr$. Thus, by definition 1 the gradient of V always has the direction of the normal to the surface $V(x, y, z) =$ constant, or

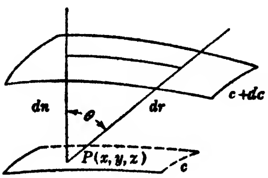


FIG. 42.

$$\nabla V = \mathbf{n}\frac{dV}{dn},$$

where $\mathbf{n}$ is a unit vector in the direction of the normal. From the last equation, the derivative of V in a general direction $\mathbf{r}$ is easily found.

Forming the scalar products of each side of the last equation with the vector $d\mathbf{r}$ we obtain

$$d\mathbf{r}\cdot\nabla V = \mathbf{n}\cdot d\mathbf{r}\frac{dV}{dn}$$

$$= dr\cos\theta\frac{dV}{dn} = dV,$$

or dividing by dr

$$\frac{dV}{dr} = \frac{d\mathbf{r}\cdot\nabla V}{dr} = \mathbf{r}_1\cdot\nabla V, \tag{242}$$

where $\mathbf{r}_1$ is a unit vector parallel to $d\mathbf{r}$. The last equation is important since it shows that the derivative of V in the direction $d\mathbf{r}$ is the projection of the gradient in that direction. This relation gives the component of ∇V in the x direction by replacing $\mathbf{r}_1$ by $\mathbf{i}$ and r by x. Thus

$$\frac{\partial V}{\partial x} = \mathbf{i}\cdot\nabla V.$$

Similarly,

$$\frac{\partial V}{\partial y} = \mathbf{j}\cdot\nabla V,$$

$$\frac{\partial V}{\partial z} = \mathbf{k}\cdot\nabla V.$$

Combining gives

$$\nabla V = \mathbf{i}\frac{\partial V}{\partial x} + \mathbf{j}\frac{\partial V}{\partial y} + \mathbf{k}\frac{\partial V}{\partial z}$$

which is the second definition of gradient. ∇V is frequently written $\sum \mathbf{i}\frac{\partial V}{\partial x}$.

The following physical examples serve to explain some applications of the gradient of a scalar. Consider the flow of heat through a solid body, and assume that it is possible to select a curvilinear section through the body at all points of which the temperature is the same. Call this section a **level surface** or an **equipotential surface.** Let V be the temperature of the body at any point $P(x, y, z)$, and let $\mathbf{q}$ be the vector representing the direction and intensity (flow per unit area) of heat flow at the point P. It is known from the theory of thermal conduction that the flow of heat will be in the direction of the greatest decrease of temperature and will have a magnitude per unit area pro-

portional to this rate of change of temperature. But ∇V is a vector with the magnitude of the greatest rate of increase of temperature and in the direction of this greatest rate of increase. Therefore, changing the sign and employing the proportionality constant k we obtain

$$\mathbf{q} = -k\nabla V.$$

This is known as Fourier's law of heat flow. The constant k is the thermal conductivity of the material. When V is known as a function of x, y, and z, the vector representing the intensity of the heat flow is readily found from the above equation.

For a second example of gradient, consider a cloud of water vapor whose density varies from point to point. Suppose that the density increases toward the geometrical center and is maximum at that point. Points at which the density has some fixed value may be selected within

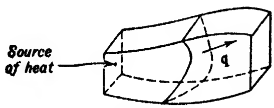


FIG. 43.

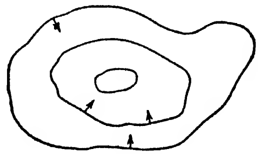


FIG. 44.

the cloud. The curvilinear surface containing all those points is a level surface or an equipotential surface for that particular density. A line drawn from a point on this surface in the direction of the greatest rate of increase of density with a length equal to this rate of increase will coincide with the vector representing the gradient at that point. Inspection of the figure shows that, in general, the gradient will vary in direction and magnitude from point to point along the same equipotential surface. It is not necessarily directed toward the point of maximum density.

74. Divergence. Three equivalent and useful definitions of the divergence of a vector function $\mathbf{F}(x, y, z)$, are the following:

1. The divergence of $\mathbf{F}(x, y, z)$, denoted by $\nabla \cdot \mathbf{F}$, is defined by the equation

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

2. The divergence of $\mathbf{F}(x, y, z)$ is defined also by the equation

$$\nabla \cdot \mathbf{F} = \mathbf{i}\cdot\frac{\partial \mathbf{F}}{\partial x} + \mathbf{j}\cdot\frac{\partial \mathbf{F}}{\partial y} + \mathbf{k}\cdot\frac{\partial \mathbf{F}}{\partial z}, \tag{243}$$

These two definitions are especially useful in establishing operator formulas. The third definition, now given, is especially useful in physical applications in connection with vector fields.

3. The divergence of $\mathbf{F}(x, y, z)$ is defined by the equation

$$\nabla \cdot \mathbf{F} \equiv \lim_{V \to 0} \frac{\displaystyle\iint_S \mathbf{n} \cdot \mathbf{F} dS}{V}, \tag{244}$$

where $\mathbf{n}$ is the exterior normal to the closed surface S whose volume is V. Integrals of the form $\iint_S \mathbf{n} \cdot \mathbf{F} dS$ have been discussed in § 70.

Thus the divergence of the vector $\mathbf{F}$ at a point is the net outward flux of $\mathbf{F}$ per unit of volume as the volume, which includes the point, is made infinitesimally small.

The equivalence of the first two definitions is established as follows:

$$\begin{aligned}
\nabla \cdot \mathbf{F} &= \mathbf{i} \cdot \frac{\partial \mathbf{F}}{\partial x} + \mathbf{j} \cdot \frac{\partial \mathbf{F}}{\partial y} + \mathbf{k} \cdot \frac{\partial \mathbf{F}}{\partial z} \\
&= \mathbf{i} \cdot \left(\mathbf{i} \frac{\partial F_x}{\partial x} + \mathbf{j} \frac{\partial F_y}{\partial x} + \mathbf{k} \frac{\partial F_z}{\partial x} \right) \\
&\quad + \mathbf{j} \cdot \left(\mathbf{i} \frac{\partial F_x}{\partial y} + \mathbf{j} \frac{\partial F_y}{\partial y} + \mathbf{k} \frac{\partial F_z}{\partial y} \right) \\
&\quad + \mathbf{k} \cdot \left(\mathbf{i} \frac{\partial F_x}{\partial z} + \mathbf{j} \frac{\partial F_y}{\partial z} + \mathbf{k} \frac{\partial F_z}{\partial z} \right) \\
&= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.
\end{aligned} \tag{245}$$

Also, beginning with the last equation, we may obtain the first. Hence, since each definition implies the other, the two are equivalent. The equivalence of the third definition with either of the first two will be shown later in this chapter after Gauss's theorem has been proved (Ex. 1, § 87).

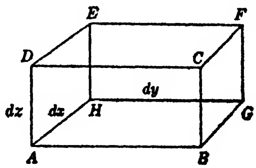


FIG. 45.

For a physical application of divergence, consider a small parallelepiped of dimensions dx, dy, dz in a mass of liquid. Assume that liquid may be flowing through all six faces of the parallelepiped. Let $\mathbf{M}$ represent the mass of the liquid flowing through a unit cross-section in unit time. (The direction of $\mathbf{M}$ is that of the velocity of flow.) Let ρ represent the density, and $\mathbf{V}$

the velocity (V_x in a direction parallel to the x-axis, V_y in a direction parallel to the y-axis, etc.); then the mass flowing through a unit area in unit time will be the density times the velocity normal to that area, or

$$\mathbf{M} = \rho\mathbf{V}.$$

The flow to the right per unit time through the face $ADEH$ is then

$$M_y dxdz = \rho V_y dxdz,$$

and the flow to the right per unit time through the face $BCFG$ is

$$\left[M_y + \frac{\partial M_y}{\partial y} dy\right] dxdz = \left[\rho V_y + \frac{\partial(\rho V_y)}{\partial y} dy\right] dxdz.$$

The net increase of fluid in the parallelepiped (due to the flow through the two faces only) is then

$$M_y dxdz - \left[M_y + \frac{\partial M_y}{\partial y} dy\right] dxdz = -\frac{\partial M_y}{\partial y} dxdydz.$$

Treating the flow through the other faces similarly and adding the results, we have, for the total increase in mass inside the parallelepiped in unit time,

$$\left(-\frac{\partial M_x}{\partial x} - \frac{\partial M_y}{\partial y} - \frac{\partial M_z}{\partial z}\right) dxdydz,$$

or dividing by $dxdydz$ we obtain for the total increase in mass per unit time per unit volume

$$-\frac{\partial M_x}{\partial x} - \frac{\partial M_y}{\partial y} - \frac{\partial M_z}{\partial z}.$$

This is recognized as $-\nabla \cdot \mathbf{M}$. Thus,

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} + \frac{\partial M_z}{\partial z} = \nabla \cdot \mathbf{M}.$$

And since the total increase in mass per unit time inside the parallelepiped is also $\frac{\partial \rho}{\partial t} dxdydz$,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{M} = -\nabla \cdot (\rho\mathbf{v}).$$

This last equation is called the **equation of continuity**, and is an expression of the principle of conservation of matter. When the liquid is incompressible, it is evident that

$$\rho = \text{constant and } \nabla \cdot \mathbf{M} = 0.$$

The divergence is the excess of the outward over the inward flow, and the convergence is the excess of the inward over the outward flow, both per unit of volume. When the divergence of a vector function of position in space vanishes in a region, the function is said to be **solenoidal** in that region.

75. Curl. Three equivalent and useful definitions of the curl of a vector function $\mathbf{F}(x, y, z)$ are the following:

1. The curl of $\mathbf{F}(x, y, z)$, denoted by $\nabla \times \mathbf{F}$, is defined by the equation

$$\nabla \times \mathbf{F} \equiv \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}. \qquad (246)$$

2. The curl of $\mathbf{F}(x, y, z)$ is defined also by the equation

$$\nabla \times \mathbf{F} \equiv \mathbf{i} \times \frac{\partial \mathbf{F}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{F}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{F}}{\partial z}. \qquad (247)$$

The equivalence of these two definitions is shown by replacing the dots by crosses in Eq. (245).

3. The final definition is very important in physical applications. The curl of $\mathbf{F}$ is a vector and hence has components in any direction. To find the component in any direction $\mathbf{s}$ at any point P choose the direction and describe a small circular area normal to the direction at the point in question. (See Fig. 46.) Form the line integral of the vector field in the conventional positive direction around the circle. (The positive direction of traversing the boundary of an area is related to the positive normal as the direction of rotation of a right-handed screw is related to its direction of advance.) Then the quotient of this line integral by the little area is, in the limit, the s component of the curl, or

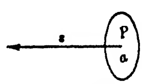


FIG. 46.

$$(\text{Curl } \mathbf{F})_s \equiv \lim_{a \to 0} \int_c \mathbf{F} \cdot d\mathbf{r} \qquad (248)$$

If three components of the curl are determined, the curl may be found by adding them vectorially.

The equivalence of this definition and those preceding is shown later. (Ex. 2 § 87).

Physically the curl may measure the tendency of the vectors of the field to run in closed loops. That is, the curl of the electric field about a point charge is zero, as may be seen by taking line integrals. But the curl of the linear velocity field of a rotating body is twice the angular velocity, and directed parallel to the axis of rotation. If the curl of a vector function of position in space vanishes everywhere in a region, the function is said to be **irrotational** in the region. The electrostatic field, the gravitational field about attracting matter, and the field of magnetic intensity in a region containing no current are irrotational vector fields. The general vector field has a curl and divergence neither of which is zero.

76. Operator Formulas. There exist a number of formulas involving vector operators. These are used in making transformations in vector fields and in deriving partial differential equations. The following are eight important ones. Of these, two are proved. The remainder can be proved in a similar manner. The symbols ϕ and $\mathbf{F}$ represent any scalar point function and any vector point function, respectively. $\mathbf{A}$ and $\mathbf{B}$ also are general vector point functions.

1. $\nabla \cdot \nabla \phi = \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + \dfrac{\partial^2 \phi}{\partial z^2} = \nabla^2 \phi.$
2. $\nabla \cdot \nabla \mathbf{F} = \dfrac{\partial^2 \mathbf{F}}{\partial x^2} + \dfrac{\partial^2 \mathbf{F}}{\partial y^2} + \dfrac{\partial^2 \mathbf{F}}{\partial z^2} = \nabla^2 \mathbf{F}.$
3. $\nabla \times \nabla \phi = 0.$
4. $\nabla \cdot (\nabla \times \mathbf{F}) = 0.$
5. $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla \cdot \nabla \mathbf{F}.$
6. $\nabla \cdot (\phi \mathbf{A}) = \phi \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \phi.$
7. $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \nabla \cdot \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{A} + \mathbf{B} \cdot \nabla \mathbf{A} - \mathbf{A} \cdot \nabla \mathbf{B}.$
8. $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}).$

EXAMPLE 1. Prove formula 3.

$$\nabla \times \nabla \phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial \phi}{\partial x} & \dfrac{\partial \phi}{\partial y} & \dfrac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \mathbf{i}\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right) + \mathbf{j}\left(\frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z}\right) + \mathbf{k}\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right) = 0.$$

EXAMPLE 2. Prove formula 5.

$$\nabla \times (\nabla \times \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left(\dfrac{\partial F_z}{\partial y} - \dfrac{\partial F_y}{\partial z}\right) & \left(\dfrac{\partial F_x}{\partial z} - \dfrac{\partial F_z}{\partial x}\right) & \left(\dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y}\right) \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\right]$$

$$+ \mathbf{j}\left[\frac{\partial}{\partial z}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\right]$$

$$+ \mathbf{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\right]$$

$$= \mathbf{i}\left[\frac{\partial^2 F_y}{\partial y \partial x} - \frac{\partial^2 F_x}{\partial y^2} - \frac{\partial^2 F_x}{\partial z^2} + \frac{\partial^2 F_z}{\partial z \partial x}\right] + \text{etc.}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)F_x\right] + \text{etc.}$$

$$= \nabla(\nabla \cdot \mathbf{F}) - \nabla \cdot \nabla \mathbf{F}.$$

EXERCISES

1. Compute by vector methods the area of the triangle whose vertices are (7, 3, 4), (1, 0, 6), and (4, 5, −2).

2. Compute both the scalar and vector products of the pairs of vectors

$$\begin{cases} \mathbf{A} = 0.6\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}, \\ \mathbf{B} = 4\mathbf{i} + 0\mathbf{j} + 10\mathbf{k}, \end{cases}$$
$$\mathbf{C} = 0.7\mathbf{i} + 9\mathbf{j} - \mathbf{k},$$
$$\mathbf{D} = \mathbf{i} + 3\mathbf{j} - 7\mathbf{k}.$$

3. Let $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. Perform the expansions $\mathbf{A}\cdot\mathbf{C}$, $\mathbf{A}\cdot\mathbf{B}$, and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$, and show that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}),$$

also

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{C}\cdot\mathbf{A})\mathbf{B} - (\mathbf{C}\cdot\mathbf{B})\mathbf{A}.$$

4. Given that

$$\mathbf{r} = \mathbf{a}\varepsilon^{T} + \mathbf{b}\varepsilon^{-}$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, show that

$$\frac{d^2\mathbf{r}}{dT^2} = \mathbf{a}\varepsilon^{T} + \mathbf{b}\varepsilon^{-T} = \mathbf{r}.$$

5. Find the derivative of the scalar point function

$$\phi \quad \frac{1}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}$$

in the direction of the vector $\mathbf{i}x + \mathbf{j}y + \mathbf{k}z$.

6. If $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, show that $\nabla \cdot \nabla(1/r) = 0$.

7. If

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \qquad (r = \sqrt{x^2 + y^2 + z^2})$$

show that

$$\nabla r \quad \frac{\mathbf{r}}{\sqrt{\mathbf{r} \cdot \mathbf{r}}}.$$

8. Prove the six unproved formulas of § 76.

II

DERIVATION OF THE PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS OR VECTOR FIELDS

The derivation of the partial differential equations of mathematical physics is little more than expressing vector relations which hold within a vector field or between vector fields. The basic relations themselves are, in general, physical relations in vector form accompanied by certain mathematical transformations resulting in the partial differential equations of § 79.

77. Some Vector Fields. There are many kinds of vector fields. In the study of heat conduction, it is known that the flow of heat is in the direction of the greatest decrease of temperature and has a magnitude per unit area proportional to the rate of change of temperature. This statement is expressed simply by the equation $\mathbf{q} = -k\nabla V$, where $\mathbf{q}$ is the heat flowing through a cross-section of unit area per unit time, the direction being that to give maximum $\mathbf{q}$; V is the temperature, a scalar function; and k is the thermal conductivity of the body. Near every mass there is a field of force called the gravitational attraction. This force of attraction at any point may be obtained by taking the gradient of a scalar point function called the gravitational potential (see § 82). Likewise, near an electrically charged body, there is the electrostatic field.

At points exterior to the charge, there exists a scalar point function, the electrostatic potential, whose gradient taken at the point $P(x,y,z)$ gives the negative of the electric intensity at that point. Near a magnetized body there is a magnetic field. The negative of the magnetic

intensity of this field is given by the gradient of the scalar magnetic potential. Within a body of flowing fluid there is a vector field or velocity field. If the curl of this field is zero then there exists a function ϕ, called the velocity potential, such that the gradient of ϕ at any point gives the negative of the velocity of the fluid at that point. In § 10 the forces acting on the body may be viewed as a limiting case of a vector field and Eq. (10) may be written $\mathbf{i}mx'' + \mathbf{i}k_dx' + \mathbf{i}kx = 0$, where $\mathbf{i}$ is the unit vector directed along the x-axis.

78. Preliminary Theorems. Before deriving the partial differential equations of mathematical physics, it is necessary to understand the very important theorems, in vector analysis, of Gauss, Stokes, and Green. In § 70, line and surface integrals involving vectors were defined and illustrated. The concept of the volume integral, $\int \mathbf{F}dv$, of a vector function is also needed. The integral $\int \mathbf{F}dv$ is defined by the equation

$$\int_{\text{Vol}} \mathbf{F}dv = \mathbf{i}\int_{\text{Vol}} F_x dv + \mathbf{j}\int_{\text{Vol}} F_y dv + \mathbf{k}\int_{\text{Vol}} F_z dv. \tag{249}$$

The three theorems of this section are the machinery by which transformations are made between line, surface, and volume integrals. Eqs. (250–252) state in symbols, respectively, Gauss's, Stokes's, and Green's theorems as follows:

$$\iiint_{\text{Vol}} \nabla\cdot\mathbf{F}dv = \iint_S \mathbf{F}\cdot d\mathbf{S}, \tag{250}$$

$$\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} = \int_c \mathbf{F}\cdot d\mathbf{r}, \tag{251}$$

$$\iiint_{\text{Vol}} (U\nabla\cdot\nabla V - V\nabla\cdot\nabla U)dv = \iint_S (U\nabla V - V\nabla U)\cdot\mathbf{n}dS \tag{252}$$

Gauss's theorem stated in words is: The volume integral of the divergence of a vector function of position in space taken over a volume is equal to the surface integral of the vector function taken over a closed surface bounding the volume. To illustrate Gauss's theorem qualitatively, consider a mass of metal within which heat is generated, say by electric current. Gauss's theorem states that the total heat flowing, in the steady state, out through the surface is equal to the

volume integral of the divergence of the heat-flow vector, which can be shown to be equal to the amount of heat generated in the solid.

Stokes's theorem is: The surface integral of the curl of a vector function of position in space taken over a surface S is equal to the line integral of the vector function taken around the periphery of the surface. A physical illustration of Stokes's theorem may be had in the magnetic field about a wire carrying a current. According to the circuital theorem the work done in carrying a unit pole around a closed path is 4π times the current enclosed by the path, or if the path lies in air ($\mu = 1$), in symbols

$$\int_c \mathbf{B}\cdot d\mathbf{r} = 4\pi I.$$

But I is equal to the surface integral of the current density $\mathbf{j}$ over any surface bounded by the closed path c,

$$\int_c \mathbf{B}\cdot d\mathbf{r} = 4\pi \int\int \mathbf{j}\cdot d\mathbf{S}.$$

The circuital theorem may also be written $\nabla \times \mathbf{B} = 4\pi\mathbf{j}$. To see that this is true it is only necessary to refer to the third definition of curl in § 75. If in Eq. (248), $\mathbf{F}$ is replaced by $\mathbf{B}$ and $\int_c \mathbf{B}\cdot d\mathbf{r}$ by $4\pi I$ we obtain

$$(\nabla \times \mathbf{B})_j = \lim_{a\to 0} \frac{\int_c \mathbf{B}\cdot d\mathbf{r}}{a} = \lim_{a\to 0} \frac{4\pi I}{a} = 4\pi j.$$

Replacing in the double integral above, $4\pi\mathbf{j}$ by $\nabla \times \mathbf{B}$ we obtain

$$\int_c \mathbf{B}\cdot d\mathbf{r} = \int_s\int \nabla \times \mathbf{B}\cdot d\mathbf{S},$$

which is Stokes's theorem. The material of this article is a statement and illustration of these theorems by means of physical examples. But these theorems depend in no way upon physical experiment. They are mathematical identities.

79. The Partial Differential Equations of Mathematical Physics. The chief partial differential equations of Mathematical Physics are the following:

a. Laplace's equation $\nabla\cdot\nabla V = 0$, which is satisfied by the functions:

1. Gravitational potential in regions unoccupied by attracting matter.

2. Electrostatic potential at points where no charge is present.
3. Magnetic potential in regions free from magnetic charges.
4. Temperature in steady state.
5. Velocity potential at points of a homogeneous non-viscous fluid moving irrotationally.
6. Electric potential in homogeneous conductors in which a current is flowing.

b. Poisson's equation $\nabla \cdot \nabla V = -e$.
c. Equation of heat conduction without sources, $a^2\nabla \cdot \nabla \theta = \theta_t$.
d. Equation of heat conduction with sources, $a^2(\nabla \cdot \nabla \theta + e) = \theta_t$.
e. Wave equation, $a^2\nabla \cdot \nabla \psi = \psi_{tt}$, and
f. $a^2(\nabla \cdot \nabla \psi + e) = \psi_{tt}$.
g. Equations of elasticity.
h. Telegraphists' equation, $a\phi_{tt} + b\phi_t + c\nabla \cdot \nabla \phi = -ce$.
i. Maxwell's field equations.
j. Euler's equation for the motion of a fluid.

The single subscript t indicates one partial differentiation with respect to time; two subscripts, partial differentiation twice. We now derive, in vector notation, some of the above equations.

80. Equation of Heat Conduction without Sources, $a^2\nabla \cdot \nabla \theta = \theta_t$. Consider the following problem: A mass of iron has been heated to a certain temperature and left to cool. What is the temperature at any point of the mass at any time t? The differential equation giving this temperature may be found from the following physical facts:

(*a*) The flow of heat will be in the direction of the greatest decrease of temperature and will have a magnitude per unit area proportional to this rate of change of temperature.

(*b*) The rate at which heat is lost by a given region of the body is the heat flux passing through the surface bounding the region.

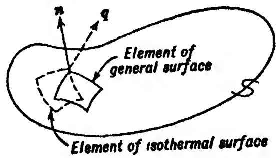


FIG. 47.

The rate of heat loss from an element of volume dv in terms of temperature θ and specific heat c is $-c\rho \dfrac{\partial \theta}{\partial t} dv$ where ρ is the density. Thus the rate of heat loss from a general region of volume V (See Fig. 47) bounded by surface S is

$$-\frac{\partial Q}{\partial t} = -\iiint_V c\rho \frac{\partial \theta}{\partial t} dv.$$

In general S is not an isothermal surface. We may also express the rate of heat loss in terms of the heat current density $\mathbf{q}$ (heat flow per unit of time per unit area normal to the flow) as

$$-\frac{\partial Q}{\partial t} = \int_S\int \mathbf{n}\cdot\mathbf{q}dS.$$

Equating these two expressions by relation (b)

$$-\int\int\int \rho c\frac{\partial \theta}{\partial t}dv = \int_S\int \mathbf{n}\cdot\mathbf{q}dS. \tag{253}$$

By means of Gauss's theorem, the last equation becomes

$$-\int\int_V\int \rho c\frac{\partial \theta}{\partial t}dv = \int\int_V\int \nabla\cdot\mathbf{q}dS.$$

Since these integrals are equal for every volume, the integrands must be equal. Hence

$$-\rho c\frac{\partial \theta}{\partial t} = \nabla\cdot\mathbf{q}.$$

But by relation (a), $\mathbf{q} = -k\nabla\theta$, where k is the thermal conductivity. The last equation then becomes

$$a^2\nabla\cdot\nabla\theta = \theta_t,$$

where $a^2 = \dfrac{k}{\rho c}$.

81. Equation of Heat Conduction with Sources. In this case, physical relations (a) and (b), § 80, still obtain, and also one additional one. Each element of the mass within the volume V may have heat generated in it by some means, for example, by an electric current. The density of strength of source e of heat is defined by the equation

$$e = \lim_{V\to 0}\frac{\text{Total heat created within } V \text{ per unit time}}{V}.$$

The additional physical relation is: the rate at which heat is emitted from the element of volume dv may be considered as consisting of two parts: first, that which is the rate of cooling the element if no source were present, namely, $-\rho c\dfrac{\partial \theta}{\partial t}dv$; and secondly, that due to the

source edv. Returning to Eq. (253) of the preceding paragraph, we write

$$\iiint_{\text{Vol}} \left(-\rho c \frac{\partial \theta}{\partial t} + e\right) dv = \iint_S \mathbf{n} \cdot \mathbf{q} dS.$$

Since this equation holds for every volume, it follows that

$$a^2(\nabla \cdot \nabla \theta + e) = \theta_t.$$

82. Concept of Potential and Theorems of General Vector Fields. It has been noted in § 77 that the gradient of a scalar point function (called various kinds of potential) gives a vector field. This leads to the definition of a potential. A potential is a scalar point function whose gradient is a vector field. In such a case, the vector field is said to possess a potential. It is by no means true that all vector fields possess a potential. The simple criterion for the existence of a potential is given by the theorem:

I. A necessary and sufficient condition that a field $\mathbf{F}$ possess a potential is that $\nabla \times \mathbf{F} = 0$. (See § 27 for the meaning of necessary and sufficient.)

To determine whether the curl of a field is zero, it is necessary to know physical facts about the field and then to apply Eq. (248). For instance, in the magnetostatic case, if the line integral $\int_c \mathbf{B} \cdot d\mathbf{r}$ is calculated around a closed path which encloses no currents, by the circuital theorem, $\int \mathbf{B} \cdot d\mathbf{r} = 0$, and consequently, by (248), curl $\mathbf{B} = 0$ in such regions. Similarly, the line integrals of the force of attraction and electric intensity, taken around closed paths, are zero for gravitational and electrostatic fields.

The concept of potential function is one of the most important in mathematical physics because, once the potential (if it exists) of the field is known, the field is determined. This raises the question, why not find the field due to the distribution of charge, current, or mass at once, and dispense with the intermediate potential? The answer is that the potential satisfies certain partial differential equations which can be integrated and hence the potential may be found with less difficulty than the field. The following table displays some of the most important potentials and their definitions.

	Definition by line integral	Definition by volume integral	Definition by partial differential equation. Solution, subject to boundary conditions of:
Newtonian potential	Negative work $=\int_\infty^r \mathbf{F}\cdot(+\,d\mathbf{r})$ per unit mass	$V=\int \frac{\mu dv}{r}$	$\nabla\cdot\nabla V = 0$, or $\nabla\cdot\nabla V = -\,4\pi\mu$.
Electrostatic potential	Work $=\int_\infty^r \mathbf{E}\cdot(-\,d\mathbf{r})$ per unit charge	$V=\int \frac{\rho dv}{r}$	$\nabla\cdot\nabla V = 0$, or $\nabla\cdot\nabla V = -\,4\pi\rho$.
Magnetic potential.	Work $=\int_\infty^r \mathbf{H}\cdot(-d\mathbf{r})$ per unit pole	$\Omega=\int \frac{\sigma dv}{r}$	$\nabla\cdot\nabla\Omega = 0$, or $\nabla\cdot\nabla\Omega = -\,4\pi\sigma$
Magnetic vector potential........	....	$\mathbf{A}=\int \frac{\mathbf{j}dv}{r}$	$\nabla\cdot\nabla\mathbf{A} = 0$, or $\nabla\cdot\nabla\mathbf{A} = -\,4\pi\mathbf{j}$
Velocity potential. .	... .	....	$\nabla\cdot\nabla\phi = 0$.
Velocity vector potential	....	$\mathbf{\Phi} = \frac{1}{2\pi}\int_{\text{vol}} \frac{\boldsymbol{\omega}}{r}\,dV$	$\nabla\cdot\nabla\mathbf{\Phi} = 0$, or $\nabla\cdot\nabla\mathbf{\Phi} = -\,2\boldsymbol{\omega}$.

In the above table:

μ = mass per unit volume,
ρ = density of charge per unit volume,
σ = pole strength per unit volume,
$\mathbf{\Phi}$ = velocity vector potential,
$\boldsymbol{\omega}$ = angular velocity of fluid = $\frac{1}{2}$ curl of linear velocity,
$\mathbf{F}$ = gravitational force,
$\mathbf{H}$ = magnetic intensity (force per unit pole) = B/μ,
ϕ = velocity potential,
$\mathbf{E}$ = electric intensity.

In the case of vector potentials the fields desired are obtained not by taking the gradient but by taking the curl of the vector potential.

From theorem I, it is evident that vector fields possessing potentials are not the most general fields since the curls of such fields have the special value zero. What then is the nature of a general vector field,

and what must be known about a general field to determine it? The answer to these two questions are theorems II and III.

II. Let **F** be a single-valued vector function which, along with its derivatives, is finite and continuous and vanishes at infinity. Then **F** can be written

$$\mathbf{F} = \nabla\phi + \nabla \times \mathbf{H},$$

where ϕ and **H** are respectively a scalar and a vector point function. This is the Helmholtz theorem in vector analysis.

III. A vector field is uniquely determined if the divergence and curl be specified, and if the normal component of the field be known over a closed surface, or if the vector vanish as $1/r^2$ at infinity. If neither of the last two conditions is satisfied, the field is determined except for an additive constant vector.

We now resume the derivation of equations.

83. Partial Differential Equations of Gravitational, Electrostatic, and Magnetostatic Fields. These derivations are based upon Gauss's law and in the case of the magnetostatic field, the circuital theorem of Ampère.

1. *Gauss's law.* In electrostatics the force between two charges q_1 and q_2 is given by the inverse square law

$$f = \frac{q_1 q_2}{r^2}.$$

The field vector **E** is defined as the force per unit charge. Gauss's law relates the surface integral of **E** over a closed surface S to the charge Q within S. For a region containing no polarized dielectric it is

$$\int_S\int \mathbf{E}\cdot d\mathbf{S} = 4\pi Q.$$

This can readily be proved from the inverse square law; in fact, it is a mathematical equivalent which is based on no further experimental evidence. Thus if a phenomenon is characterized by Coulomb's inverse square law, as the magnetostatic and gravitational fields are, Gauss's law also holds. For the magnetostatic field **E** in Gauss's law is replaced by **H**, the force per unit pole, and Q, by the number of unit poles enclosed. For the gravitational field, **E** is replaced by **F**, the force per unit mass, and Q by $-M$, the *negative* of the total mass enclosed. The negative sign occurs because the force between masses is attraction whereas that between like charges or like poles is repulsion.

The circuital theorem. The line integral $\int_c \mathbf{H}\cdot d\mathbf{r}$ of the mag-

netic intensity $\mathbf{H}$, due to a current, taken around any closed path c encircling a conductor is equal to $4\pi I$, where I is the total current flowing in the conductor.

By means of 1 and 2 above, most of the fundamental laws governing gravitational, electrostatic, and magnetostatic fields are quickly obtained.

(*a*) *Gravitation.* In gravitational fields Gauss's law is

$$\int_S\int \mathbf{F}\cdot d\mathbf{S} = -\,4\pi M,$$

where M is the total mass enclosed. The last equation may be written

$$\int_S\int \mathbf{F}\cdot d\mathbf{S} = -\,4\pi\int_{\text{Vol}} \mu dv,$$

where μ is the mass density. Applying Gauss's theorem (250), we have

$$\int_{\text{Vol}} \nabla\cdot\mathbf{F}dv = -\,4\pi\int_{\text{Vol}} \mu dv.$$

Since the last equation holds for every volume, it follows that the integrands are equal, that is,

$$\nabla\cdot\mathbf{F} = -\,4\pi\mu. \tag{254}$$

By applying the definition of curl (248) to a gravitational field, which obeys the inverse square law, it can be shown that $\nabla \times \mathbf{F} = 0$ everywhere. By theorem I, § 82, a potential V exists such that $\nabla V = \mathbf{F}$. Hence (254) can be written

$$\nabla\cdot\nabla V = -\,4\pi\mu. \tag{255}$$

This is Poisson's equation. It holds at all points occupied by matter. At points free from attracting matter $\mu = 0$, and Poisson's equation becomes Laplace's equation

$$\nabla\cdot\nabla V = 0. \tag{256}$$

Eqs. (255) and (256) are the important partial differential equations of gravitational theory.

(*b*) *Magnetostatics.* Replacing $\mathbf{E}$ by $\mathbf{H}$ and Q by $\int_{\text{Vol}} \sigma\, dv$ in Gauss's law, and repeating the reasoning immediately preceding (254), we have

$$\nabla\cdot\mathbf{H} = 4\pi\sigma. \tag{257}$$

The quantity σ is the pole strength per unit volume. By the circuital theorem and the definition of curl (248), it follows that in non-current-carrying regions

$$\nabla \times \mathbf{H} = 0.$$

Hence by theorem I, § 82, potential function Ω exists in non-current-carrying regions such that $\nabla\Omega = -\mathbf{H}$. Hence (257) becomes

$$\nabla \cdot \nabla\Omega = -4\pi\sigma. \tag{258}$$

At points devoid of magnetic poles the last equation becomes

$$\nabla \cdot \nabla\Omega = 0. \tag{259}$$

In current-carrying regions, by the circuital theorem, $\nabla \times \mathbf{H} \neq 0$, and consequently no scalar potential Ω exists.

(*c*) *Electrostatics.* By retracing the steps employed in (*a*) of this article, it follows that

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho, \\ \nabla \times \mathbf{E} &= 0, \\ \nabla V &= -\mathbf{E}, \\ \nabla \cdot \nabla V &= -4\pi\rho. \end{aligned} \tag{260}$$

The quantities ρ and V are defined in § 82. So far, the electrostatic charges considered in the application of Gauss's law have been free charges. Gauss's law as stated above holds only if there is no dielectric medium within the closed surface. Suppose now, in addition to free charges, there is within S a dielectric containing bound charges which are influenced by an electric field. The field causes the positive atom cores and negative electrons of an atom to be displaced from their equilibrium (normal) position. The result is that the atom forms a **dipole.** The product of either charge of a dipole by the separating distance is called the magnitude of the electric moment of the dipole. If the direction is taken from the negative charge to the positive charge, the product of this unit vector by the magnitude of the moment is called the **electric moment,** a vector quantity. The polarization $\mathbf{P}$ of a dielectric is defined to be the total electric moment per unit volume. It can be shown that the **polarization** of the atoms of dielectric is equivalent to a mean charge per unit volume of $-\nabla \cdot \mathbf{P}$. Hence Gauss's theorem becomes

$$\int_S \mathbf{E} \cdot d\mathbf{S} = 4\pi\left(Q - \int_V \nabla \cdot \mathbf{P}\, dv\right)$$

$$= 4\pi(Q - \int_S \mathbf{P}\cdot d\mathbf{S}),$$

or

$$\int_S (\mathbf{E} + 4\pi\mathbf{P})\cdot d\mathbf{S} = 4\pi Q. \tag{261}$$

The quantity $\mathbf{E} + 4\pi\mathbf{P}$ is called the **electric displacement** and is denoted by $\mathbf{D}$. Hence Gauss's law for all charges within S is

$$\int_S \mathbf{D}\cdot d\mathbf{S} = 4\pi Q.$$

Again proceeding as in (*a*), we have

$$\nabla\cdot\mathbf{D} = 4\pi\rho \tag{262}$$

instead of

$$\nabla\cdot\mathbf{E} = 4\pi\rho.$$

We are now in a position to derive Maxwell's field equations.

84. Maxwell's Equations. For the derivation of these equations, in general form, there are needed: (*a*) certain results of § 83, (*b*) the experimental results of Faraday and Ampère, (*c*) vector relations, and (*d*) Maxwell's generalization.

(*a*) *Results of § 83.* In Eqs. (257–262) electrostatic and electromagnetic units are employed. The most important of these equations, if written in Heaviside-Lorentz rational units (to eliminate the 4π's), are

$$\begin{aligned} \nabla\cdot\mathbf{D} &= \rho \\ \nabla\cdot\mathbf{B} &= 0, \\ \nabla\cdot\mathbf{E} &= \rho, \\ \int_c \mathbf{H}\cdot d\mathbf{r} &= \mathbf{I}. \end{aligned} \tag{263}$$

($\mathbf{I}$ is in rational electromagnetic units.)

Eqs. (263) hold for steady currents and stationary electrostatic charges and stationary circuits. It is natural to expect the existence of a set of simultaneous partial differential equations describing the more general electromagnetic configurations, that is, those configurations or systems in which there are moving circuits and charges not at rest. These equations are the well-known **field equations.**

(*b*) *Experiments of Faraday and Ampère.* In 1831 Faraday discovered the fact that, whenever the magnetic flux through a closed single-turn circuit varies, there is induced in the circuit an electro-

motive force whose magnitude is equal to the time rate of decrease of flux. The direction of the electromotive force is related to the direction of flux through the circuit as shown in Fig. 48. If the electromotive force is induced in a conductor a current flows.

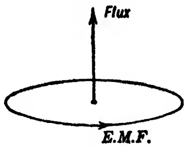


FIG. 48.

Ampère first obtained experimentally the results upon which the theorem stated in § 83 is based.

(*c*) *Mathematical expression of Faraday's and Ampère's laws.* The electromotive force e around the closed curve C formed by a circuit is defined by the line integral

$$e = \int_c \mathbf{E} \cdot d\mathbf{r},$$

taken around the curve. By Stokes's theorem,

$$\int_c \mathbf{E} \cdot d\mathbf{r} = \int_S \int \nabla \times \mathbf{E} \cdot d\mathbf{S}, \tag{264}$$

where S is a cap (surface) whose periphery is the circuit or curve C. Faraday's experimental result, expressed in vector form, is

$$\int_S \int (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\frac{1}{c}\frac{d}{dt}\int_S \int \mathbf{B} \cdot d\mathbf{S} = -\frac{1}{c}\int \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S},$$

or

$$\int \int (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\frac{1}{c}\int \int \dot{\mathbf{B}} \cdot d\mathbf{S},$$

where the dot over a quantity indicates partial time differentiation, and c is a constant of proportionality, equal to the velocity of light, necessary in this system of units. Since the last equation is true for every surface S, it follows that

$$\nabla \times \mathbf{E} = \quad \frac{1}{c}\dot{\mathbf{B}}. \tag{265}$$

Eq. (265) is Faraday's law in differential form. Ampère's circuital theorem, in vector notation, is

$$\int_c \mathbf{H} \cdot d\mathbf{r} = \int_S \int \mathbf{j} \cdot d\mathbf{S},$$

where S is a cap whose periphery is C. By Stokes's theorem, we also have

$$\int_c \mathbf{H} \cdot d\mathbf{r} = \int_S \int \nabla \times \mathbf{H} \cdot d\mathbf{S}.$$

Consequently,

$$\int_S\int \nabla \times \mathbf{H}\cdot d\mathbf{S} = \int_S\int \mathbf{j}\cdot d\mathbf{S},$$

or by the reasoning preceding Eq. (265)

$$\nabla \times \mathbf{H} = \mathbf{j}. \tag{266}$$

Eq. (266) is Ampère's law in differential form. If it is assumed that Gauss's law is valid for variable fields as well as for electrostatic and magnetostatic fields, we then have the four equations:

$$\left.\begin{aligned} \nabla\cdot\mathbf{D} &= \rho, \\ \nabla\cdot\mathbf{B} &= 0, \\ \nabla\times\mathbf{E} &= -\frac{1}{c}\dot{\mathbf{B}}, \\ \nabla\times\mathbf{H} &= \mathbf{j} = \frac{\rho\mathbf{v}}{c}, \end{aligned}\right\} \tag{267}$$

where $\mathbf{v}$ is the drift velocity of charge of density ρ.

(*d*) *Maxwell's generalization.* Maxwell noted that Eqs. (267) are inconsistent with the equation of continuity of charge. The equation of continuity of mass, $\frac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{M}$, was derived in § 74. If ρ denotes charge per unit volume and $\mathbf{v}$ its velocity, the equation of continuity, in electromagnetic theory, becomes

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\mathbf{v}). \tag{268}$$

This equation merely states that the time rate of increase of charge in any region is equal to the excess of charge flowing in, per unit time, over that flowing out. All experimental evidence indicates that the law of continuity holds, that electricity is neither created nor destroyed.

The contradiction between Eq. (268) and the first and last of (267) is seen as follows. Taking the divergence of $\nabla \times \mathbf{H} = \frac{\rho\mathbf{v}}{c}$, we have

$$\nabla\cdot\left(\frac{\rho\mathbf{v}}{c}\right) = \nabla\cdot\nabla\times\mathbf{H} = 0, \tag{269}$$

or

$$\nabla\cdot\left(\frac{\rho\mathbf{v}}{c}\right) = 0.$$

But (268) gives

$$\nabla \cdot (\rho \mathbf{v}) = -\frac{\partial \rho}{\partial t}.$$

Moreover, if the first of (267) is differentiated with respect to time, there is

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \dot{\mathbf{D}},$$

and from the value of $\frac{\partial \rho}{\partial t}$ in the equation of continuity

$$\nabla \cdot (\rho \mathbf{v}) = -\nabla \cdot \dot{\mathbf{D}}. \tag{270}$$

Equations (269) and (270) do not agree. Accordingly, Maxwell revised Ampère's law as follows. Let the total current consist of a convection current $\frac{\rho \mathbf{v}}{c}$ and a displacement current $\frac{\dot{\mathbf{D}}}{c}$. The $\mathbf{j}$ in equation (266) is then replaced by $\frac{1}{c}(\rho \mathbf{v} + \dot{\mathbf{D}})$, and Ampère's equation as revised by Maxwell becomes

$$\nabla \times \mathbf{H} = \frac{1}{c}(\rho \mathbf{v} + \dot{\mathbf{D}}). \tag{271}$$

If the divergence of (271) is taken, Eq. (270) is obtained. But (270) is a consequence of (268) and the first of (267). Thus the equation of continuity is satisfied if system (267) be replaced by the equations

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -\frac{1}{c}\dot{\mathbf{B}}, \\ \nabla \times \mathbf{H} &= \frac{1}{c}(\rho \mathbf{v} + \dot{\mathbf{D}}). \end{aligned} \tag{272}$$

These are the field equations of Maxwell.

If the currents are steady the $\dot{\mathbf{D}} = 0$ and (272), in this special case, reduce to (267).

In regions devoid of charge and current equations (272), since $\mathbf{B} = \mu \mathbf{H}$ and $\mathbf{D} = k\mathbf{E}$, become

$$
\begin{aligned}
\nabla \cdot \mathbf{E} &= 0, \\
\nabla \cdot \mathbf{H} &= 0, \\
\nabla \times \mathbf{E} &= -\frac{\mu \dot{\mathbf{H}}}{c}, \\
\nabla \times \mathbf{H} &= \frac{k}{c}\dot{\mathbf{E}}.
\end{aligned}
\tag{273}
$$

The constants μ and k are respectively the permeability and dielectric constant of the space for which (273) are valid.

The nature of the solution of these equations is discussed in § 86, and the equations are solved in Vol. II, Chap. III, for configurations of charge and current of great industrial importance.

85. Euler's Equation for the Motion of a Fluid. The physical principle on which Euler's equation for the motion of a perfect fluid rests is nothing more or less than Newton's law of motions explained in § 9. But since we shall need to consider the acceleration of a mass of fluid, it is first necessary to derive the mathematical expression for this acceleration.

A perfect fluid is one which cannot support a tangential stress. Let $\mathbf{v}$ be the linear velocity of an element of fluid at the point $P(x, y, z)$ at the time t, and $\mathbf{v} + d\mathbf{v}$ its velocity at $Q(x, y, z)$ near P at

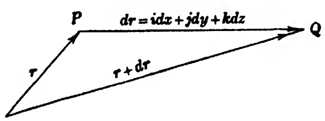


FIG. 49.

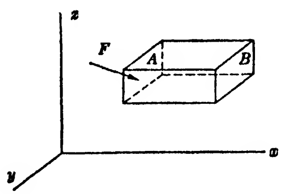


FIG. 50.

the time $t + dt$. Obviously, $\mathbf{v}$ is a function of the four variables x, y, z, and t. Hence, by the expression for a total differential,

$$
\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{v}}{\partial z}\frac{dz}{dt} + \frac{\partial \mathbf{v}}{\partial t} \equiv \nabla \mathbf{v} \cdot \frac{d\mathbf{r}}{dt} + \frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \cdot \nabla \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}, \tag{274}
$$

where

$$
\nabla \mathbf{v} = \mathbf{i}\frac{\partial \mathbf{v}}{\partial x} + \mathbf{j}\frac{\partial \mathbf{v}}{\partial y} + \mathbf{k}\frac{\partial \mathbf{v}}{\partial z}.
$$

Evidently $\dfrac{d\mathbf{v}}{dt}$ is the acceleration of the fluid at P.

Let the mass of fluid in an infinitesimal rectangular parallelepiped $dxdydz$ be $\rho dxdydz$. Suppose that this infinitesimal mass is acted upon by a field of force $\mathbf{F}$ per unit mass of the fluid, and also by the pressure p due to the remainder of the fluid. The pressure is normal to the faces of the parallelepiped. Let the force due to the pressure p acting on the face A be $\mathbf{i}pdydz$. Then the force on B is $-\mathbf{i}\left(p + \frac{\partial p}{\partial x}dx\right)dydz$. The net force acting to the right due to the pressure is $-\mathbf{i}\frac{\partial p}{\partial x}dxdydz$. Let the x-component of the force F be F_x. The total force acting to the right then is $\mathbf{i}\left(\rho F_x - \frac{\partial p}{\partial x}\right)dxdydz$. By Eq. (274), the acceleration of the infinitesimal mass $dxdydz$ is $\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right)$. Denote the x-component of this acceleration by $\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right)_x$. Since the sum of the components of all the forces acting, including the inertial reaction, is zero, we have

$$\mathbf{i}\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right)_x \rho dxdydz = \mathbf{i}\left(\rho F_x - \frac{\partial p}{\partial x}\right)dxdydz. \tag{275}$$

We have two similar equations for the components of the forces along the y- and z-axes. Adding these two equations to Eq. (275), there results,

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \mathbf{F} - \frac{1}{\rho}\nabla p. \tag{276}$$

This is Euler's equation of fluid motion. If $\mathbf{F}$ is derivable from a potential Ω, (276) becomes

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \nabla\Omega - \frac{1}{\rho}\nabla p. \tag{277}$$

86. Nature of the Solution of Partial Differential Equations. In § 7 the meaning of the solution of an nth-order ordinary differential equation was explained. To solve such an equation means to find a function of the independent variable which satisfies the differential equation and which contains n arbitrary constants. The boundary (initial) conditions consist of specified values of the function and of the first $(n - 1)$ derivatives for some definite value of the independent variable.

The solution of a partial differential equation is similar except for

the boundary conditions. The number of independent variables is at least two. To construct a solution it is then necessary to find a function of the independent variables, which when substituted in the partial differential equation satisfies it. In general, there is an infinitude of such functions, but only one of these will also satisfy the boundary conditions. The boundary conditions in two great classes of partial differential equations (including most of those of mathematical physics) are as follows. The value of the solution must reduce to a prescribed function over a surface boundary, or the normal derivative of the solution must reduce to a prescribed function over a boundary or satisfy certain other conditions.

In solving a partial differential equation which holds throughout a region under consideration the equation in vector form is transformed into a scalar equation. The coordinate system is chosen such that, when one of the independent variables is set equal to a certain constant, the boundary of the region is obtained. For example, if the region, in which $\nabla^2 V = 0$ is valid, is a rectangular parallelepiped, then Laplace's equation is written

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

and the plane $x = x_0$ is a portion of the boundary of the region. If the region is a cylinder, ellipsoid, sphere, or tore then the coordinate system chosen must be respectively cylindrical, ellipsoidal, spherical, or toroidal orthogonal coordinates. These coordinates are much used in the solution of partial differential equations.

At the beginning of Vol. II, Chap. III, on the solution of partial differential equations, a general method is given for transforming a partial differential equation into one of the infinitely many systems of orthogonal coordinates.

In § 18, a method of solving systems of simultaneous ordinary differential equations was explained. By manipulation one differential equation was obtained in one of the dependent variables of the system. A system of partial differential equations is handled in a similar manner. This is illustrated in the derivation, from Maxwell's field equations, of the equation of the propagation of electromagnetic waves.

87. The Partial Differential Equations of Electromagnetic Waves. The type of general problem described by Eqs. (272) and (273) is as follows. A system of moving charges and varying currents exists in a varying configuration. It is required to find the **H** and **E** fields at

time t. Once $\mathbf{H}$ and $\mathbf{E}$ are obtained, any question can be answered regarding the system. For example, it can be shown easily (Ex. 3) that the force on a charge q moving in an electromagnetic field is a function of $\mathbf{H}$ and $\mathbf{E}$, namely,

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right), \tag{278}$$

where $\mathbf{B} = \mu\mathbf{H}$.

As in § 18, Eqs. (273) are, in general, not solved directly but replaced by two other equations; one in $\mathbf{H}$ alone, the other in $\mathbf{E}$ alone. This is done as follows. Taking the curl of the third equation of (273), we have

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\mu}{c}\nabla \times \dot{\mathbf{H}} = -\frac{\mu}{c}\frac{\partial}{\partial t}\nabla \times \mathbf{H}. \tag{279}$$

The partial derivative with respect to the time of the last equation of (273) gives

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{H}) = \frac{k}{c}\frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{280}$$

Substituting the value of $\frac{\partial}{\partial t}(\nabla \times \mathbf{H})$ from (280) in (279), there is

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\mu k}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2},$$

or, since $\nabla \cdot \mathbf{E} = 0$,

$$\nabla^2 \mathbf{E} = +\frac{\mu k}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

(See § 76, formula 5.) Similarly,

$$\nabla^2 \mathbf{H} = \frac{\mu k}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}. \tag{281}$$

Eqs. (281) are the equations sought. Applications of these equations in the solution of problems of value are found in Ref. 6.

EXERCISES

1. Show that the first and third definitions of divergence given in § 74 are equivalent. By Gauss's theorem

$$\int_{\text{vol}} \nabla \cdot \mathbf{V} dv = \int_S \mathbf{V} \cdot d\mathbf{S}.$$

By the theorem of the mean from the calculus

$$\int f(x, y, z) dv = f(x_0, y_0, z_0) \int dv,$$

where (x_0, y_0, z_0) is some point of the volume. Applying the last equation to the one preceding it

$$\int_S \mathbf{V}\cdot d\mathbf{S} = \int_{\text{vol}} \nabla\cdot\mathbf{V}dv = \nabla\cdot\mathbf{V}(x_0, y_0, z_0)\int_{\text{vol}} dv = \nabla\cdot\mathbf{V}(x_0, y_0, z_0)\text{ vol.}$$

$$\nabla\cdot\mathbf{V} = \lim_{\text{vol}\to 0}\frac{\int_S \mathbf{V}\cdot d\mathbf{S}}{\text{vol}}.$$

2. Show, by means of Stokes's theorem and the method of Ex. 1, that the first and third definitions of curl in § 75 are equivalent.

3. Prove that the force on a charge q moving in an electromagnetic field is

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right).$$

4. Prove Gauss's, Green's, and Stokes's theorems.

III

VECTOR FIELDS

(Vector Magnetic Theory)

In the design of electrical apparatus, it is frequently necessary to know the magnetic flux density not only in the neighborhood of the winding but also within the conductors themselves. The flux density can be computed, at points exterior to the conductors, from the scalar magnetic potential. This is not the case at points within the conductor because the scalar magnetic potential does not exist inside current-carrying regions. The flux density can, however, be computed from a mathematical expression called the vector magnetic potential. One of the purposes of this section is to explain those portions of vector magnetic theory that pertain to the vector potential. Since a number of important engineering papers have been written employing the vector potential, a second purpose of this section is to furnish the background for these papers and to correlate their important concepts. Such will furnish the starting point for further investigations in vector potential of more and more complicated regions.

As in Section II of this chapter certain mathematical proofs are reserved for exercises. These are found at the end of the section.

88. Experimental Basis of Magnetic Theory. There are a number of consistent logical systems of magnetic theory, and the subject is so well developed that many approaches are possible. An approach is desired which is primarily an engineering one and also such that the

idea of vector potential and related concepts may be reached as quickly as possible. Because electrical engineering is greatly concerned with the interaction of electric currents we shall base this section upon Ampère's fundamental law. The following relationships hold only for steady currents or, what amounts to the same thing, configurations of moving charges which may vary from point to point, but which at any given location do not, on the average, vary in time. The discussion of magnetic vector potential is further restricted here to the case of non-magnetic media, i.e., regions characterized by unit permeability. Ampère's experiments were made with complete circuits, or at most dealt with the force between one complete circuit and the movable element of another. The following law for the force between circuit elements is equivalent to his results, provided it be used to find the force between a complete circuit and a circuit element

$$d\mathbf{F} = \frac{II'}{r^3}\, d\mathbf{s} \times (d\mathbf{s}' \times \mathbf{r}), \tag{282}$$

where $d\mathbf{F}$ is the force exerted by the element of length ds', in which current I' flows, on the element ds in which current I flows. The current is positive if positive charge flows in the direction of the element. The vector $\mathbf{r}$ is drawn from $d\mathbf{s}'$ to $d\mathbf{s}$ as shown in Fig. 51. This and succeeding equations are written for quantities expressed in c.g.s. electromagnetic units.

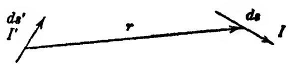


FIG. 51.

89. Force between Moving Charges. Let us express (282) in terms of moving charges, making use of the fact that the current at any point in a conductor is the charge passing that point per unit of time. If an element ds contains n charges e moving with drift speed v,

$$I = \frac{nev}{ds},$$

or

$$Id\mathbf{s} = ne\mathbf{v}.$$

Thus a single charge e moving with a velocity $\mathbf{v}$ is equivalent to a current element $Id\mathbf{s}$ if

$$Id\mathbf{s} = e\mathbf{v}.$$

Thus from (282) the force between moving charges e and e' is

$$\mathbf{F} = \frac{e\mathbf{v}}{r^3} \times (e'\mathbf{v}' \times \mathbf{r}),$$

where $\mathbf{r}$ is the vector distance from e' to e. This may be expressed in terms of the field vector $\mathbf{B}$ due to e' as

$$\mathbf{F} = e\mathbf{v} \times \mathbf{B}, \tag{283}$$

where

$$\mathbf{B} = \frac{e'\mathbf{v}' \times \mathbf{r}}{r^3}. \tag{284}$$

The magnetostatic force on e due to a number of moving charges $e'_1, e'_2 \ldots e'_n$ is still given by (283) if now

$$\mathbf{B} = \sum_{i=1}^{i=n} \frac{e'_i\mathbf{v}'_i \times \mathbf{r}_i}{r_i^3}. \tag{285}$$

90. Vector Magnetic Potential. In the same way that it is convenient in electrostatic problems to define a potential function from which to calculate electric intensities, it is convenient in magnetic problems to set up a vector magnetic potential $\mathbf{A}$, from which to calculate the magnetic flux densities $\mathbf{B}$; the vector magnetic potential exists both within and exterior to current-carrying regions.

The vector potential is defined so that

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{286}$$

By the second definition of curl, Eq. (247), the vector identity

$$\nabla \times \frac{e_i\mathbf{v}_i}{r_i} = e_i\mathbf{v}_i \times \frac{\mathbf{r}_i}{r_i^3}$$

is readily established. From the last equation and the definition of $\mathbf{B}$, (285), it follows that

$$\mathbf{B} = \sum \nabla \times \frac{e_i\mathbf{v}_i}{r_i} = \nabla \times \sum \frac{e_i\mathbf{v}_i}{r_i}.$$

Now if we define the magnetic vector potential $\mathbf{A}$ by the equation

$$\mathbf{A} = \sum \frac{e_i\mathbf{v}_i}{r_i}, \tag{287}$$

then

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Thus when $\mathbf{A}$ has been determined for a circuit, the flux density $\mathbf{B}$ is obtained by a routine process, namely, by the application of the curl operator, Eq. (246).

91. Integral Definition of Vector Potential. Since the vector potential is to play such a fundamental rôle in magnetic theory, it is important to know its properties. It has been defined in Eq. (287)

for groups of charge, but its use, of course, depends upon expressing it for ordinary bodies carrying currents. The form of the vector potential is perfectly general and may be applied to any sort of moving configuration of charge. However, if we here restrict the motions of charges to translation only—that is, if we consider only elementary currents excluding current whirls—then (287) may be written in terms of current density as

$$\mathbf{A} = \int_{\text{Vol}} \frac{\mathbf{j}dv}{r}, \tag{288}$$

where the integrand is current density in the volume element divided by the distance of the volume element from the point at which **A** is to be found. The definition (288) is applicable only to *finite* bodies. In § 93 this definition is modified to take care of straight conductors of infinite length.

Since $\mathbf{B} = \nabla \times \mathbf{A}$, it follows from Eq. (4), § 76, that $\nabla \cdot \mathbf{B} = 0$. This implies that tubes of magnetic flux are always closed on themselves or that flux lines are closed loops. It is true also that, if there is no " heaping up " of charge in the body, $\nabla \cdot \mathbf{A} = 0$. Consequently, similar closed loops for **A** can be constructed.

92. Partial Differential Equation Definition of Vector Potential. The vector potential of a finite body may be found by definition (288) provided the volume integration can be performed. It is frequently easier to obtain the vector potential by the solution of a partial differential equation.

In Vol. II, Chap. III, the following important theorem in the theory of partial differential equations is proved. Theorem: A solution, continuous with its first derivatives and vanishing at infinity as $1/r$, of the equation

$$\nabla \cdot \nabla V = -4\pi\rho \tag{289}$$

is given by

$$V_P = \int \frac{\rho' dv'}{r}, \tag{290}$$

where V is a potential function corresponding to the density ρ. If ρ is density of electric charge, V is electric potential; if ρ is mass density, V is gravitational potential.

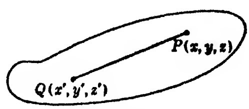


FIG. 52.

The notation is made clear by Fig. 52, in which Q is the moving point of the element of integration and P the fixed point at which the potential V is to be found. The density at Q is denoted by $\rho'(x', y', z')$.

If Eqs. (289) and (290) are written in rectangular coordinates, we have

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\, 4\pi\rho(x, y, z),$$

$$V_P = \int\int\int \frac{\rho'(x', y', z')dx'dy'dz'}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}}.$$

If V and ρ are replaced in (289) and (290) by **A** and **j**, and certain boundary conditions taken care of, then the following equivalent definition of the vector potential for finite bodies may be written. A function satisfying, in regions of unit permeability, the conditions:

$$\left.\begin{aligned} &(a) \quad \nabla^2\mathbf{A} = -\, 4\pi\mathbf{j}, \\ &(b) \quad \mathbf{A} \text{ is continuous}, \\ &(c) \quad \frac{\partial \mathbf{A}}{\partial n_1} + \frac{\partial \mathbf{A}}{\partial n_2} = 0, \\ &(d) \quad \mathbf{A} \text{ is regular at } \infty, \end{aligned}\right\} \qquad (291)$$

is the vector potential defined by (288) in case there is no surface magnetization at the boundary of the conductor or in the neighboring medium. The $\frac{\partial \mathbf{A}}{\partial n}$ is the derivative of **A** in any direction n, and n_1 and n_2 are respectively the interior and exterior normal directions to a surface. If there is surface magnetization, condition (c) must be modified. Condition (c), as stated above, means that the normal derivative is continuous at boundaries between substances both of unit permeability, such as air and copper. It is not continuous at an iron boundary. (See Ex. 2 § 96.)

93. Vector Potential for Infinite Conductors. As will be shown below, the vector potential **A** at point P due to the current in an infinitely long straight conductor is given by

$$= -\int_S\int \mathbf{j} \log r^2 ds, \qquad (292)$$

where S is the cross-section in the plane perpendicular to the axis of the conductor including point P, r is the distance from P to the element ds at which the current density is **j**. This naturally raises the question, how is Eq. (292) related to the definitions of (288) and (291)? Definitions (288) and (291) are most used in engineering. (See Refs. 35–36 at end of text.)

It is the purpose of this article to show the derivation of (292) from (288). Suppose that an infinitely long wire coincides with the z-axis as in Fig. 53. By a wire is meant a conductor of infinitesimal cross-section, say ds. Let the current density in the wire be $\mathbf{j}$ (which is in the z direction), then the current in the wire is $\mathbf{i} = \mathbf{j}ds$. Consider a segment (finite length) of the wire. Let the ends of the segment be C_1 and C_2. Find the vector potential at the point $P(x, z)$ by (288). The potential at P is

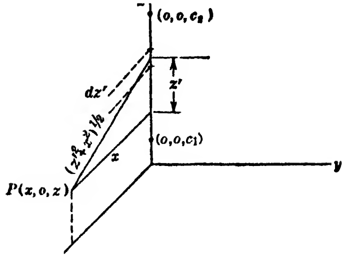


FIG. 53.—Vector Potential of Infinite Wire.

$$\mathbf{A} = \int \frac{\mathbf{i}dz'}{r} = \mathbf{i}\int_{(C_1-z)}^{(C_2-z)} \frac{d}{(z'^2 + x^2)^{\frac{1}{2}}} = \mathbf{i} \log \left[z' + \sqrt{z'^2 + x^2}\right]\Big|_{(C_1-z)}^{(C_2-z)}$$

or

$$\mathbf{A} = \mathbf{i} \log \frac{[x^2 + (C_2 - z)^2]^{\frac{1}{2}} + C_2 - z}{[x^2 + (C_1 - z)^2]^{\frac{1}{2}} + C_1 - z}.$$

The equipotential surfaces about the segment can be shown to be ellipsoids with C_1 and C_2 as foci. In connection with the study of fields about conductors, we desire for simplicity equipotential surfaces which are cylinders. Consequently, we let $C_1 \to -\infty$ and $C_2 \to \infty$, and the ellipsoids approach cylinders, provided the point $P(x, z)$ and the direction of the line remain fixed. As C_1 and C_2 become infinite in the manner indicated, the expression for $\mathbf{A}$ becomes

$$\mathbf{A}_\infty = \lim_{\substack{C_1 \to -\infty \\ C_2 \to \infty}} \left[\mathbf{i} \log \frac{[x^2 + (C_2 - z)^2]^{\frac{1}{2}} + C_2 - z}{[x^2 + (C_1 - z)^2]^{\frac{1}{2}} + C_1 - z}\right] = \infty.$$

This is an unsatisfactory result. Hence we shall not take Eq. (288) to be the definition of potential when the body extends to infinity. We observe that, if the body is enclosed in a finite volume, the zero of the potential, as given by Eq. (288), is at infinity. We now lead up to a definition of potential in the case in which we are interested. By rationalizing the denominator in the last fraction, we have for the potential of the finite segment,

$$\mathbf{A} = \mathbf{i}\Big\{\log \frac{1}{x^2} + \log\left[\sqrt{x^2 + (C_2 - z)^2} + (C_2 - z)\right] + \log\left[\sqrt{x^2 + (C_1 - z)^2} - (C_1 - z)\right]\Big\}.$$

This value of $\mathbf{A}$ is the potential according to Eq. (288) at the point $P(x, z)$ of a finite wire segment. Subtract from this value $\mathbf{A}$ a constant $\mathbf{K}$ (i.e., a constant so far as the point $P(x, z)$ is concerned, but not constant with reference to C_1 and C_2). For example, let $\mathbf{K}$ be the potential at the point (a, b) due to the finite wire. The potential at $P(x, z)$ due to the infinite wire then is *defined* to be the previous value minus $\mathbf{K}$, or now

$$\begin{aligned}\mathbf{A} = \mathbf{i}\,\Big\{\log\frac{1}{r^2} &+ \log\left[\sqrt{x^2 + (C_2 - z)^2} + (C_2 - z)\right] \\ &+ \log\left[\sqrt{x^2 + (C_1 - z)^2} - (C_1 - z)\right] \\ - \log\frac{1}{a^2} &- \log\left[\sqrt{a^2 + (C_2 - b)^2} + (C_2 - b)\right] \\ &- \log\left[\sqrt{a^2 + (C_1 - b)^2} - (C_1 - b)\right]\Big\}.\end{aligned}$$

Taking the limit as $C_1 \to -\infty$, $C_2 \to \infty$, we obtain

$$\mathbf{A} = \mathbf{i}\log\frac{1}{x^2} - \mathbf{i}\log\frac{1}{a^2}.$$

When $x = a$, i.e., on a cylinder of radius a about the wire, $\mathbf{A} = 0$. The zero of $\mathbf{A}$ is then somewhat arbitrary, depending upon the value of a. If for simplicity a is taken to be unity

$$\mathbf{A} = \mathbf{i}\log\frac{1}{x^2} = -\,\mathbf{i}\log x^2.$$

This definition, which is the usual one, of the potential of an infinite wire is a natural one, since it is the potential (slightly modified) of a segment of a wire as the segment becomes infinite. The modification imposed changes the zero of the vector potential from infinity to the distance a. If we do not choose a zero point of the potential $\mathbf{A}$, then

$$\mathbf{A} = \mathbf{i}\log\frac{1}{x^2} + \text{an arbitrary constant}.$$

To obtain the vector potential of an infinite conductor, we sum, by means of the integral, the vector potential of the wires of infinitesimal cross-section. Hence,

$$\mathbf{A} = \int\!\!\int_S -\,\mathbf{j}\log r^2\,dS,$$

where r, Fig. 54, is the distance from the element dS to the point $P(r)$.

In finding the vector potential of an infinite straight wire carrying a current of density j, the vector potential is seen to be independent of the z-coordinate of the point P. (See Fig. 53.) Consequently if Eq. (291) is written in rectangular coordinates the term $\frac{\partial^2 A}{\partial z^2}$ is zero and $\nabla^2 \mathbf{A} = -4\pi \mathbf{j}$ becomes

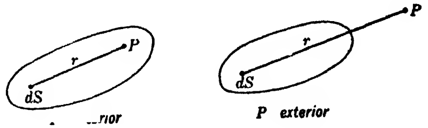


FIG. 54.—Vector Potential of Infinite Conductor.

$$\frac{\partial^2 \mathbf{A}}{\partial x^2} + \frac{\partial^2 \mathbf{A}}{\partial y^2} = -4\pi \mathbf{j}. \tag{293}$$

94. Engineering Examples. Let it be required to find, both by evaluating the integral in Eq. (292) and by solving the partial differential equation (291), the vector potential due to an infinitely long straight conductor carrying a current of density j. The cross-section of the wire is circular, of radius a. Let the zero of the vector potential be on the unit circle, the axis of the wire being the z-axis.

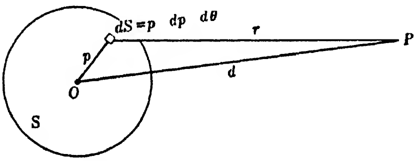


FIG. 55.—Exterior Vector Potential.

First method. Let $\mathbf{A}_e$ denote the vector potential at an exterior point. Then

$$\mathbf{A}_e = -\int_S\int \mathbf{j} \log r^2 dS$$

$$= -\mathbf{j}\int_0^a\int_0^2 \log\,(d^2 + p^2 - 2pd\cos\theta)p\,dp\,d\theta$$

$$= -\mathbf{j}\int_0^a\int_0^{2\pi}\left[\log d^2 + \log\left(1 - \frac{2p}{d}\cos\theta + \frac{p^2}{d^2}\right)\right]p\,dp\,d\theta$$

$$= -\mathbf{j}\pi a^2 \log d^2 \quad \text{(see Ex. 1)},$$

or

$$= -2\mathbf{I}\log d,$$

where $\mathbf{I}$ is the total current flowing in the conductor. If P is interior to the cross-section, then

$$\mathbf{A}_i = -\mathbf{j}\left[\int_0^d\int_0^{2\pi}\log r^2 dS + \int_d^a\int_0^{2\pi}\log r^2 dS\right].$$

The value of the first integral is $-2\pi\mathbf{j}d^2 \log d$. The value of the second integral is

$$-\ 2\pi a^2\mathbf{j} \log a + 2\pi d^2\mathbf{j} \log d + \pi a^2\mathbf{j} - \pi d^2\mathbf{j}.$$

(See Ex. 1.) Thus

$$\mathbf{A}_i = \pi\mathbf{j}(a^2 - d^2) - 2\pi a^2\mathbf{j} \log a.$$

Second method. The same result is now obtained by solving the differential equation $\nabla\mathbf{A}^2 = -\ 4\pi\mathbf{j}$ or $\nabla^2\mathbf{A} = 0$ according as the point P is interior or exterior to the current-carrying region. Since the conductor is circular in cross-section, we express the differential equation in cylindrical coordinates. By the rules of the calculus for change of independent variables, Eq. (293) becomes

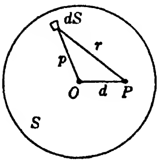


FIG. 56. Interior Vector Potential.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \mathbf{A}}{\partial r}\right) = -\ 4\pi\mathbf{j}. \qquad \text{(See Ex. 4.)} \qquad (294)$$

We have found $\mathbf{A}$ to have the direction of $\mathbf{j}$ (along the z-axis). Consider only the z component of $\mathbf{A}$. Outside the conductor $\mathbf{j}$ is, of course, zero and the equation to be solved is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_{z_e}}{\partial r}\right) = 0.$$

Integrating with respect to r, we have

$$r\frac{\partial A_{z_e}}{\partial r} = C.$$

Integrating again,

$$A_{z_e} = C \log r + k. \qquad (295)$$

For points inside the conductor Eq. (294) must be solved. Since $\mathbf{A}$ and $\mathbf{j}$ have the same direction, Eq. (294) may be written

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_{z_i}}{\partial r}\right) = -\ 4\pi j.$$

The current density $\mathbf{j}$ is constant over the cross-section. Integrating, there is

$$r\frac{\partial A_{z_i}}{\partial r} = -\ 2\pi r^2 j + C_1.$$

Integrating again

$$A_{z_i} = -\ \pi r^2 j + C_1 \log r + k_1. \qquad (296)$$

The constants C, k, C_1, and k_1 of Eqs. (295–296) are determined subject to conditions (*b*) and (*c*) of Eqs. (291), and subject to the two additional conditions that **A** vanish at $r = 1$ and that **A** remain finite at the origin, i.e., on the axis of the wire. **A** is known to be finite at the origin by definition (292). Applying the last condition first, it follows that $C_1 = 0$. If $a < 1$, we must apply the condition that $\mathbf{A} = 0$ at $r = 1$ to Eq. (295). Hence $k = 0$. Applying the condition that **A** (or A_z in this case) is continuous at the boundary $r = a$, the values of A_z given by Eqs. (295–296) must be equal for $r = a$. That is,

$$C \log a = - \pi j a^2 + k_1. \tag{297}$$

Condition (*c*), namely,

$$\left.\frac{\partial \mathbf{A}}{\partial n_1}\right|_{r=a} + \left.\frac{\partial \mathbf{A}}{\partial n_2}\right|_{r=a} = 0,$$

or

$$\frac{\partial A_{z_e}}{\partial r} = \frac{\partial A_{z_i}}{\partial r},$$

gives the equation

$$\left.\frac{\partial (C \log r)}{\partial r}\right|_{r=a} = \left.\frac{\partial}{\partial r}(- r^2 j\pi + k_1)\right|_{r=a},$$

or

$$C = - 2\pi a^2 j. \tag{298}$$

From (297) and (298) it follows that

$$k_1 = \pi j a^2 - 2\pi j a^2 \log a.$$

Substituting in Eqs. (295–296) the values of the arbitrary constants which have been determined, we have

$$\mathbf{A}_e = - \pi a^2 \mathbf{j} \log r^2,$$

$$\mathbf{A}_i = \pi a^2 \mathbf{j} - \pi r^2 \mathbf{j} - 2\pi a^2 \mathbf{j} \log a.$$

These expressions for A_e and A_i agree, of course, with those obtained by the definition of Eq. (292).

95. Additional Vector Relations in Vector Magnetic Theory. Vector identity (5) of § 76 is

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla \cdot \nabla \mathbf{F}.$$

It was noted in § 91 that $\nabla \cdot \mathbf{A} = 0$. From § 92 we have

$$\nabla^2 \mathbf{A} = \nabla \cdot \nabla \mathbf{A} = - 4\pi \mathbf{j}.$$

From these three relations it follows that

$$\nabla \times (\nabla \times \mathbf{A}) = 4\pi\mathbf{j}. \tag{299}$$

But $\nabla \times \mathbf{A} = \mathbf{B}$. Hence

$$\nabla \times \mathbf{B} = 4\pi\mathbf{j}. \tag{300}$$

96. Summary. To obtain the flux density in current-carrying regions (and non-current-carrying regions) compute the vector potential by either of the definitions (288), (291), or (292). Take the curl of A. This is B the flux density. If the coordinates used are rectangular coordinates, it is only necessary to apply Eq. (246). If the coordinates are cylindrical, spherical, or any other curvilinear coordinates, it is necessary to employ the expression for curl in curvilinear coordinates developed in Vol. II, Chap. III.

EXERCISES

1. Show that

$$\int_0^\pi \log\left(1 - \frac{2\rho}{d}\cos\theta + \frac{\rho^2}{d^2}\right)d\theta = 0 \text{ for}\left(\frac{\rho}{d}\right) < 1$$

$$= \pi \log\left(\frac{\rho}{d}\right)^2 \text{ for}\left(\frac{\rho}{d}\right)^2 \geqq 1.$$

First from

$$1 - 2r\cos\theta + r^2 = (1 - re^{i\theta})(1 - re^{-i\theta}) = r^2\left(1 - \frac{e^{i\theta}}{r}\right)\left(1 - \frac{e^{-i\theta}}{r}\right)$$

prove

$$\log(1 - 2r\cos\theta + r^2) = -2\left(r\cos\theta + \frac{r^2}{2}\cos 2\theta + \frac{r^3}{3}\cos 3\theta + \dots\right)$$

and

$$\log(1 - 2r\cos\theta + r^2) = +2\log r - 2\left(\frac{\cos\theta}{r} + \frac{\cos 2\theta}{2r^2} + \frac{\cos 3\theta}{3r^3} + \dots\right).$$

2. For a discussion of the behavior of **B** and **A** at boundary regions between substances of different permeability read:

(*a*) "Fundamental Theory of Flux Plotting," Stevenson, *General Electric Review*, Vol. XXIX, November, 1926.

(*b*) *The Electromagnetic Field*, Mason and Weaver, p. 208.

3. Obtain the vector potential for an infinite conductor of square cross-section and constant current density j.

4. Show that $\dfrac{\partial^2 A}{\partial x^2} + \dfrac{\partial^2 A}{\partial y^2} = -4\pi j$ in rectangular coordinates reduces, by change of dependent variables, to

$\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial A}{\partial r}\right) = -4\pi j$ in cylindrical coordinates, if A is not a function of θ.

IV

DYADICS

This introduction to dyadics is directly preparatory for the study of the General Electric Company text, *An Analysis of Synchronous Machines*, and for the derivation of the equations of elasticity. This brief treatment is divided into three parts:

(*a*) Certain formal definitions.
(*b*) A number of necessary theorems.
(*c*) Elementary applications.

97. Definitions. The symbol **ab** (i.e., two vectors placed in juxtaposition) is called a **dyad.** An algebraic sum of such terms (**ab** + **cd** + **ef** + . . .) is called a **dyadic.** Since any dyadic can be reduced to the algebraic sum of three (or less) dyads, it is necessary to discuss dyadics only of the form **ab** + **cd** + **ef.** A dyadic is a mathematical operator having no physical significance in itself, but the important operations performed by this operator have physical significance.

98. Digression from Definitions to a Physical Example. To see in one case, at least, how dyadics arise, consider the following problem. Let a deformable body be subject to a homogeneous strain. Let it be required to express the displacement of a general point P as a function of the vector position of the point prior to the strain. A body is **deformable** if its particles are capable of displacement relative to each other. Such a relative displacement is called a **strain.** Let $P(x, y, z)$ denote the location of a particle of the body prior to the strain, and $P'(x', y', z')$ the location of the same particle after the strain takes place. In case of a **homogeneous** strain, the coordinates x', y', z' of P' are **linearly** expressible with finite scalar constant coefficients in terms of the coordinates x, y, z of P. By reference to Fig. 57, this definition is expressed by means of the equations

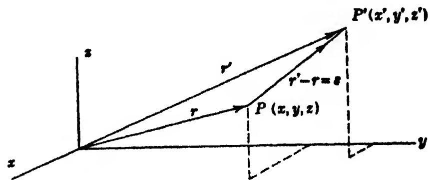


FIG. 57.

$$
\begin{aligned}
x' &= a_{11}x + a_{12}y + a_{13}z, \\
y' &= a_{21}x + a_{22}y + a_{23}z, \\
z' &= a_{31}x + a_{32}y + a_{33}z.
\end{aligned}
\tag{301}
$$

Evidently from the figure

$$\begin{aligned} \mathbf{r} &= \mathbf{i}x + \mathbf{j}y + \mathbf{k}z, \\ \mathbf{r}' &= \mathbf{i}x' + \mathbf{j}y' + \mathbf{k}z'. \end{aligned} \tag{302}$$

In view of Eqs. (301), Eq. (302) is

$$\mathbf{r}' = \mathbf{i}(a_{11}x + a_{12}y + a_{13}z) + \mathbf{j}(a_{21}x + a_{22}y + a_{23}z) + \mathbf{k}(a_{31}x + a_{32}y + a_{33}z) \tag{303}$$

$$= \mathbf{i}\mathbf{a}_1\cdot\mathbf{r} + \mathbf{j}\mathbf{a}_2\cdot\mathbf{r} + \mathbf{k}\mathbf{a}_3\cdot\mathbf{r}, \tag{304}$$

where

$$\left.\begin{aligned} \mathbf{a}_1 &= \mathbf{i}a_{11} + \mathbf{j}a_{12} + \mathbf{k}a_{13}, \\ \mathbf{a}_2 &= \mathbf{i}a_{21} + \mathbf{j}a_{22} + \mathbf{k}a_{23}, \\ \mathbf{a}_3 &= \mathbf{i}a_{31} + \mathbf{j}a_{32} + \mathbf{k}a_{33}. \end{aligned}\right\} \tag{305}$$

Eq. (304) may be written in the form

$$\mathbf{r}' = (\mathbf{i}\mathbf{a}_1 + \mathbf{j}\mathbf{a}_2 + \mathbf{k}\mathbf{a}_3)\cdot\mathbf{r}, \tag{306}$$

where the right side of (306) is only a symbol denoting the same as the right side of (304). We have thus obtained in (306) a new symbol, the dyadic. If from each side of (304) the vector $\mathbf{r}$ is subtracted, then

$$\begin{aligned} \mathbf{s} = \mathbf{r}' - \mathbf{r} &= \mathbf{i}\mathbf{a}_1\cdot\mathbf{r} + \mathbf{j}\mathbf{a}_2\cdot\mathbf{r} + \mathbf{k}\mathbf{a}_3\cdot\mathbf{r} - \mathbf{i}\mathbf{i}\cdot\mathbf{r} - \mathbf{j}\mathbf{j}\cdot\mathbf{r} - \mathbf{k}\mathbf{k}\cdot\mathbf{r} \\ &= (\mathbf{i}\mathbf{a}_1 + \mathbf{j}\mathbf{a}_2 + \mathbf{k}\mathbf{a}_3 - \mathbf{i}\mathbf{i} - \mathbf{j}\mathbf{j} - \mathbf{k}\mathbf{k})\cdot\mathbf{r}. \end{aligned} \tag{307}$$

The algebraic sum of the six dyads on the right side of the equation is a dyadic. Thus (307) may be written

$$= \psi\cdot\mathbf{r}, \tag{308}$$

where ψ is a dyadic. Thus the symbol $\psi\cdot$ operating on the position vector $\mathbf{r}$ yields a displacement vector $\mathbf{s}$. We now return to the subject of definitions and lay down such further ones that from them we may establish theorems which in turn are useful in the application of dyadics to physical problems.

99. Definitions Resumed. The dyadics $\mathbf{ab} + \mathbf{cd} + \mathbf{ef}$ and $\mathbf{ba} + \mathbf{dc} + \mathbf{fe}$ are **conjugates.**

The **dot product** of a dyadic $\psi = \mathbf{ab} + \mathbf{cd} + \mathbf{ef}$ into a vector $\mathbf{r}$ is defined by the equation

$$\psi\cdot\mathbf{r} = \mathbf{ab}\cdot\mathbf{r} + \mathbf{cd}\cdot\mathbf{r} + \mathbf{ef}\cdot\mathbf{r} = \Sigma\, \mathbf{ab}\cdot\mathbf{r}.$$

Likewise $\mathbf{r}\cdot\psi = \Sigma\, \mathbf{r}\cdot\mathbf{ab}$. From these two definitions, it follows that in general $\psi\cdot\mathbf{r} \neq \mathbf{r}\cdot\psi$.

The **cross product** of a dyadic ψ into a vector $\mathbf{r}$ is defined by the equation $\psi \times \mathbf{r} = \Sigma\, \mathbf{ab} \times \mathbf{r}$. Two dyadics ψ and ϕ are said to be equal if and only if

$$\mathbf{r}\cdot\phi = \mathbf{r}\cdot\psi \tag{309}$$

and

$$\phi\cdot\mathbf{r} = \psi\cdot\mathbf{r} \tag{310}$$

for all values of $\mathbf{r}$. In the dot product $\mathbf{r}\cdot\psi$ the dyadic is called a **post factor**; in $\psi\cdot\mathbf{r}$ the dyadic ψ is called a **prefactor**. If a dyadic ϕ is such that $\phi\cdot\mathbf{r} = \mathbf{r}$ and $\mathbf{r}\cdot\phi = \mathbf{r}$ for all values of $\mathbf{r}$, then ϕ is an idemfactor, usually denoted by I. If all the vectors in the dyadic $\phi = \mathbf{ab} + \mathbf{cd} + \mathbf{ef}$ are expressed in terms of their $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ components and the results expanded there results

$$\begin{aligned}\phi = {} & a_{11}\mathbf{ii} + a_{12}\mathbf{ij} + a_{13}\mathbf{ik} \\ & + a_{21}\mathbf{ji} + a_{22}\mathbf{jj} + a_{23}\mathbf{jk} \\ & + a_{31}\mathbf{ki} + a_{32}\mathbf{kj} + a_{33}\mathbf{kk},\end{aligned}$$

where the a's are scalar constants. This form of ϕ is the **nonion** form. The nonion form of the idemfactor is $I = \mathbf{ii} + \mathbf{jj} + \mathbf{kk}$.

100. Theorems. The theorems desired are written as the following equations:

$$\begin{aligned}&\text{I.} && \psi\cdot(\mathbf{r}_1 + \mathbf{r}_2) = \psi\cdot\mathbf{r}_1 + \psi\cdot\mathbf{r}_2,\\ &\text{II.} && \mathbf{r}\cdot(\psi_1 + \psi_2) = \mathbf{r}\cdot\psi_1 + \mathbf{r}\cdot\psi_2,\\ &\text{III.} && (\mathbf{a} + \mathbf{b})\mathbf{c} = \mathbf{ac} + \mathbf{bc},\\ &\text{IV.} && \phi\cdot(\mathbf{r} \times \mathbf{s}) = (\phi \times \mathbf{r})\cdot\mathbf{s},\\ &\text{V.} && \mathbf{a} \times \mathbf{r} = (I\cdot\mathbf{a}) \times \mathbf{r} = (I \times \mathbf{a})\cdot\mathbf{r} = -\,\mathbf{r}\cdot(I \times \mathbf{a}).\end{aligned}$$

To establish theorem I, apply the definition of the dot product and obtain

$$\begin{aligned}\psi\cdot(\mathbf{r}_1 + \mathbf{r}_2) &= \Sigma\mathbf{ab}\cdot(\mathbf{r}_1 + \mathbf{r}_2)\\ &= \Sigma\mathbf{ab}\cdot\mathbf{r}_1 + \Sigma\mathbf{ab}\cdot\mathbf{r}_2\\ &= \psi\cdot\mathbf{r}_1 + \psi\cdot\mathbf{r}_2,\end{aligned}$$

which completes the proof. We leave the proof of II, III, and IV as exercises but establish V. By the definition of idemfactor I, it is true that $I\cdot\mathbf{a} = \mathbf{a}$. Obviously $\mathbf{a} \times \mathbf{r} = (I\cdot\mathbf{a}) \times \mathbf{r}$. To see that

$(I\cdot\mathbf{a}) \times \mathbf{r} = (I \times \mathbf{a})\cdot\mathbf{r}$ it is necessary only to reduce both expressions to $\mathbf{a} \times \mathbf{r}$. Now

$$(I\cdot\mathbf{a}) \times \mathbf{r} = \mathbf{a}\times \mathbf{r}$$

and

$$(I \times \mathbf{a})\cdot\mathbf{r} = \Sigma\,(\mathbf{ii} \times \mathbf{a})\cdot\mathbf{r} = \Sigma\mathbf{i}(\mathbf{i} \times \mathbf{a})\cdot\mathbf{r}$$
$$= \Sigma\,\mathbf{i}(\mathbf{i}\cdot\mathbf{a}) \times \mathbf{r} = I\cdot\mathbf{a} \times \mathbf{r} = \mathbf{a} \times \mathbf{r}.$$

This concludes the proof.

101. Applications. Both the theory and applications of dyadics are as extensive as the theory and applications of vectors. Many analogous theorems hold in the two subjects. We are interested in only three elementary applications. The first, with reference to elasticity, has already been given. The second is concerned with the rotational property of dyadics.

(*a*) *Rotational dyadics.* In Fig. 58, if $\mathbf{a}$ is a unit vector then the operator $\mathbf{a} \times$ applied to the vector $\mathbf{r}$ perpendicular to $\mathbf{a}$ turns $\mathbf{r}$ through a right angle. This follows from the definition of a cross-product. But by theorem *V*, $\mathbf{a} \times \mathbf{r} = (I \times \mathbf{a})\cdot\mathbf{r}$. Hence $(I \times \mathbf{a})\cdot$ is an operator which rotates the vector $\mathbf{r}$ through a right angle about the

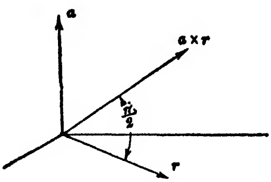


Fig. 58.

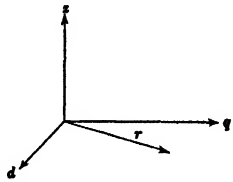


Fig. 59.—Direct and Quadrature Axes.

line $\mathbf{a}$. If again $(I \times \mathbf{a})\cdot$ be applied to the vector $(I \times \mathbf{a})\cdot\mathbf{r}$ the vector $(I \times \mathbf{a})\cdot\mathbf{r}$ is turned through 90 degrees or $\mathbf{r}$ is then turned through 180 degrees. If $\mathbf{r}$ is parallel to $\mathbf{a}$, then $(I \times \mathbf{a})\cdot$ annihilates $\mathbf{r}$ since in this case $\mathbf{r} \times \mathbf{a} = 0$. If $\mathbf{r}$ is neither parallel nor perpendicular to $\mathbf{a}$, the $(I \times \mathbf{a})\cdot$ annihilates that component of $\mathbf{r}$ parallel to $\mathbf{a}$ and rotates through 90 that component of $\mathbf{r}$ perpendicular to $\mathbf{a}$. Let the unit vectors $\mathbf{d}$, $\mathbf{q}$, and $\mathbf{z}$ be directed along the x, y, and z axes respectively. (This notation is useful in synchronous machine theory where $\mathbf{q}$ is in the direction of the quadrature and $\mathbf{d}$ in the direction of the direct axis.) The idemfactor then is $I = \mathbf{dd} + \mathbf{qq} + \mathbf{zz}$. Let $\mathbf{r}$ be any vector perpendicular to the z-axis.

Then $(I \times \mathbf{z})\cdot\mathbf{r} = [(\mathbf{dd} + \mathbf{qq} + \mathbf{zz}) \times \mathbf{z}]\cdot\mathbf{r} = (\mathbf{qd} - \mathbf{dq})\cdot\mathbf{r}$. But by theorem V, we have

$$(I \times \mathbf{z})\cdot\mathbf{r} = \mathbf{z} \times \mathbf{r},$$

where $\mathbf{z}\times$ turns $\mathbf{r}$ through 90 degrees. Since the operator $\sqrt{-1}$ performs the same operation on complex numbers, it follows that

$$\sqrt{-1} \text{ is equivalent to } (\mathbf{qd} - \mathbf{dq}),$$

each in its own system of representation. This is *a relation of frequent occurrence in some treatments of synchronous-machine theory.*

(*b*) *Impedance as a dyadic.* Let the unit vectors **i**, **j**, **k** be taken along the x, y, z axes. Electrical impedance, $r + x\sqrt{-1}$ in complex number notation, may then be written

$$\begin{aligned}\mathbf{z} &= \mathbf{i}r + \mathbf{j}x \\ &= r(\mathbf{ii} + \mathbf{jj})\cdot\mathbf{i} + (\mathbf{ji} - \mathbf{ij})\cdot\mathbf{i}x \\ &= rI\cdot\mathbf{i} + x(\mathbf{k}\times I)\cdot\mathbf{i} \\ &= [rI + x(\mathbf{k}\times I)]\cdot\mathbf{i},\end{aligned}$$

where I is the idemfactor $\mathbf{ii} + \mathbf{jj}$.

EXERCISES

1. If g is conductance and b susceptance, express admittance as a dyadic operator.
2. Given that

$$\psi_d = g_d(p)e_d - x_d(p)i_d,$$

$$\psi_q = g_q(p)e_q - x_q(p)i_q,$$

and

$$\mathbf{e}_d = e_d\mathbf{d},\ \mathbf{e}_q = e_q\mathbf{q},\ \mathbf{i}_d = i_d\mathbf{d}$$

$$\mathbf{e} = \mathbf{e}_d + \mathbf{e}_q,\ \mathbf{i} = \mathbf{i}_d + \mathbf{i}_q, \text{ etc.},$$

where **d** and **q** are unit vectors in line with the direct and quadrature axes, respectively Show that

$$\boldsymbol{\psi} = g(p)\cdot\mathbf{e} - x(p)\cdot\mathbf{i},$$

where

$$g(p) = g_d(p)\mathbf{dd} + g_q(p)\mathbf{qq},$$

$$x(p) = x_d(p)\mathbf{dd} + x_q(p)\mathbf{qq}.$$

CHAPTER IV

HEAVISIDE'S OPERATIONAL CALCULUS

The Heaviside operational calculus is, to both electrical and mechanical engineers, one of the most valuable branches of mathematics. There are, at present, three main approaches to this subject. The first is by means of line integrals in the complex plane, the second by Laplace's integral equation, and the third by operator-experimental processes. A general approach is desired so that the student has not only a tool by which a few particular well-known exercises can be solved, but also an *instrument of research by which he can make difficult investigations.* The line integral method satisfies this requirement. This approach has the disadvantage that the student is delayed in the study of the operational calculus until certain theorems on line integrals are understood, but the subsequent rapid progress, due to more powerful methods, compensates for this delay. The only knowledge presupposed for understanding this chapter is the definition of a definite integral from the calculus, the elementary principles of circuits explained in §§ 19–20, and the theory of determinants found in §§ 21–27.

I

PRELIMINARY MATHEMATICAL PRINCIPLES—LINE INTEGRALS IN THE COMPLEX PLANE

The line integral theorems required are: Cauchy's first and second integral theorems, Laurent's theorem or expansion, and the residue theorem. But before understanding these theorems, in preparation for the study of the operational calculus, let us first answer the natural questions: what kind of engineering problems does Heaviside's operational calculus solve, and what is the history of the development of this subject concerning which there has been so much controversy.

102. Some Engineering Problems Solvable by Operational Calculus. The operational calculus is of greatest use in obtaining transient responses of electrical and thermal circuits and of mechanical systems to suddenly impressed voltages, heat densities, and forces, respectively. It will, however, give also the steady-state response. The value of

this calculus is quickly displayed by the general statement of engineering problems which it readily solves.

(*a*) *Electric circuits with concentrated (lumped) parameters.* Let there be given a linear network of n meshes (branches) with concentrated circuit parameters. Let the n meshes be coupled in any or all of the ways described in § 20. Suppose the n meshes are in a state of equilibrium, that is, no currents flowing or charges existing. It is required to find the response (currents in each branch) of the network when a constant voltage E is suddenly applied in any mesh.

Suppose that, instead of a constant voltage, a voltage which is a function of the time is suddenly applied. Find the response.

Finally let n variable voltages be applied, one in each branch of the network. Again, find the response.

If, instead of applied voltages, we have suddenly applied currents in the meshes of a network the operational calculus will yield the voltages induced across any element of the network at time t subsequent to the application of the currents.

In certain cases the response can be calculated even if the network is not in equilibrium when the voltages are impressed.

(*b*) *Transmission line.* Consider a transmission line possessing distributed inductance, resistance, capacitance, and leakage. Suppose in addition there is a concentrated impedance at both the sending and receiving ends of the line. Let it be required to find the current at any time t after a voltage has been suddenly impressed at the sending end of the line.

(*c*) *Linear heat flow.* Consider a flat wall consisting of alternate layers of metal and insulating material. Let the wall be of sufficient area that a flow of heat within the wall may be considered to flow in a direction perpendicular to the wall. Suppose that a definite amount of heat is given to each unit area per unit time. Find the temperature at any point of the wall at any time t.

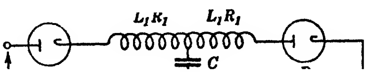


FIG. 60.—Thyratron Inverter.

(*d*) *Condenser-type Thyratron inverter.* A constant voltage E is applied to the Thyratron inverter circuit as shown in Fig. 60. The circuit is arranged so that one and only one tube is firing at any time. Obtain transient and steady-state load current. (See Ref. 6.)

(*e*) *Seismographs.* The operational calculus is a natural tool for the investigation and interpretation of the motions of seismographs.

In this case the suddenly impressed quantities are either velocities or accelerations. The reader is referred to Vol. II.

103. Historical Note. Oliver Heaviside lived during the period 1850–1925. One of his many contributions to knowledge of electrical phenomena was the operational calculus with which his name is always associated. His efforts in this field were but slowly appreciated, partly because his results were not obtained in a mathematically rigorous way. The formulas derived by him yielded correct results in almost every case, but his justification of them was not pleasing to certain of his mathematical and engineering contemporaries. However, in 1916, T. J. I'A. Bromwich in England and K. W. Wagner in Germany placed the Heaviside operational calculus on a mathematically rigorous foundation by means of the theory of functions of a complex variable. In 1925, John Carson also gave rigorous proofs, by means of integral equations, of Heaviside's methods. In 1927, H. W. March showed that Bromwich's integral is a solution of Carson's integral equation. These papers form the framework of the rigorous theory of the operational calculus. References to these and other important papers on the subject are found at the end of the text. *It should, of course, be remembered that Heaviside was the discoverer or inventor of the subject which bears his name, and that the subsequent rigorous proofs, although of great value, are in a rather definite sense only an improvement.*

If the student has as much knowledge of the theory of functions as is given in Vol. II, Chap. IV, it will be time saved to omit §§ 104–114 and proceed at once with the operational calculus in § 115. On the other hand, no knowledge of function theory is presupposed in this chapter, and the account of line integrals in §§ 104–114 is ample for the understanding of Bromwich's rigorous results. The remainder of this section is devoted to the explanation of the requisite theorems on line integrals in the complex plane.

104. Complex Numbers and Functions. Although a knowledge of the properties of and elementary operations with complex numbers such as representation by Argand's diagram, addition, multiplication, division, extraction of roots, and De Moivre's theorem for n a positive integer is presupposed, it is wise to review these subjects briefly. Just as all real numbers may be represented by points along the x-axis, so complex numbers (i.e., numbers of the form $x + iy$ where x and y are real numbers and $i = \sqrt{-1}$) may be represented by points in a plane. The old x-axis and y-axis of analytic geometry become respectively the axis of reals and the axis of imaginaries. The point $x + iy$ is then plotted as shown in Argand's diagram, Fig. 61. The real num-

bers $\rho = +\sqrt{x^2 + y^2}$ and $\theta = \arctan y/x$ are called respectively the modulus and the amplitude of the complex number $z = x + iy$. Evidently, from the figure,

$$z = \rho(\cos\theta + i\sin\theta).$$

In elementary algebra it is shown that if $z_1 = \rho_1(\cos\theta_1 + i\sin\theta)$ and $z_2 = \rho_2(\cos\theta_2 + i\sin\theta_2)$, then

$$z_1 z_2 = \rho_1\rho_2\,[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)],$$

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2}\,[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$$

It is also shown that

$$z^n = [\rho(\cos\theta + i\sin\theta)]^n = \rho^n(\cos n\theta + i\sin n\theta),$$

where n is any positive integer. The last relation is De Moivre's theorem.

The complex quantity $z = x + iy$, where x and y are independent real variables and $i = \sqrt{-1}$, is called a complex variable. Any expression of the form $U(x, y) + iV(x, y)$ is a function of $z = x + iy$, if, when x and y are given, at least one value of U and V are known. If no restrictions are placed upon U and V, the study of functions of the form $U(x, y) + iV(x, y)$ becomes nothing more than the study of pairs of real functions, the i fulfilling no purpose. If, however, proper restrictions are placed upon U and V, there results a subclass of functions which possess many of the properties of real functions and permit the development of a calculus called the theory of functions of a complex variable. These restrictions appear later.

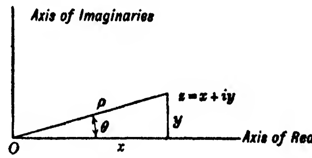


FIG. 61.

In the study of continuous, single-valued, real functions, $y = y(x)$, defined in the interval $x_1 \leqq x \leqq x_2$, the graph of $y = y(x)$ is a continuous curve. As the independent variable x ranges over the interval $x_1 \leqq x \leqq x_2$, the point $P(x, y)$ describes the curve and the ordinate y of the point $P(x, y)$ gives the value of the function.

In the study of continuous single-valued complex functions, such as $W = f(z)$, for each value of the independent variable $z = x + iy$ corresponding to a point in the z-plane, there is a value of $W = U + iV$ corresponding to a point in the W-plane. For example, suppose

$W = z^2$. Let z have the value $1 + \sqrt{3}i$, then $W = (1 + \sqrt{3}i)^2 = -2 + 2\sqrt{3}i$, and we have the representation as follows:

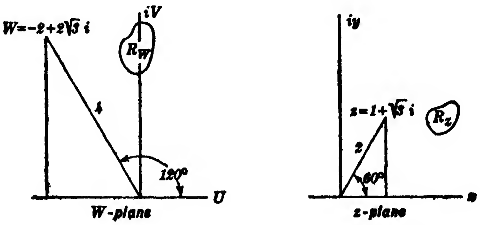


FIG. 62.

And in general if z takes on values within some region such as R_z, for each value of z there will be a corresponding value of W lying in the region R_W in the W-plane.

A function $W = f(z)$ is said to be continuous at the point z if to every positive number ε another positive number δ can be found, such that

$$|f(z + h) - f(z)| < \varepsilon$$

whenever $|h| < \delta$. This is the same as the definition of continuity of a real function except that the absolute value of h replaces the positive quantity h. Obviously if $f(z) = U(x, y) + iV(x, y)$ is a continuous function, then both U and V are continuous functions.

A **regular arc** is defined to be a set of points such that

1. $x = f(t)$, $y = g(t)$ where t is a parameter and $t_1 \leqq t \leqq t_2$.
2. $f(t)$, $g(t)$, $f'(t)$, and $g'(t)$ are single-valued and continuous for all values of t for which $t_1 \leqq t \leqq t_2$.
3. $[f'(t)]^2 + [g'(t)]^2 \neq 0$.

A **regular curve** consists of a finite number of regular arcs which are joined end to end. A **regular curve** is said to be **closed** if $f(t_1) = f(t_2)$ and $g(t_1) = g(t_2)$. A portion of the plane R is said to be **connected** if every pair of points in R may be joined by a regular curve possessing only points of R. A **region** is defined to be a connected portion of a plane. A closed region, that is, a connected portion of the plane including its boundary, is called a **connex**. A region is **simply connected** if any closed curve in it encloses only points of the region.

105. Definition of a Line or Curvilinear Integral in the Complex Plane. Before beginning the study of curvilinear integrals in the complex plane, we review certain principles of the calculus. Let

$y = f(x)$ be a real function. Divide the interval $a \leqq x \leqq b$ into n parts of lengths $\Delta x_1, \Delta x_2, \Delta x_3 \ldots \Delta x_n$. Let δ be as great as any Δx_i for $i = 1, 2, 3 \ldots n$. The positive number δ is called the **norm** of $\Delta x_1, \Delta x_2, \ldots \Delta x_n$. Let ξ_i be a point in Δx_i, either an interior or an end point. If

$$\lim_{\substack{\delta \to 0 \\ n \to \infty}} [f(\xi_1)\Delta x_1 + \cdots f(\xi_n)\Delta x_n] = \int_a^b f(x)dx$$

exists, this limit is the definite integral of $f(x)$ from a to b. This limit exists for a very large class of functions.

The definite integral in the complex plane is defined in a similar manner. Let the curve MNP, Fig. 63, be divided into n parts by the points $z_1, z_2, \ldots z_n$. The lengths $(\Delta z)_1, (\Delta z)_2, \ldots (\Delta z)_n$, where $(\Delta z)_i = z_i - z_{i-1}$, are directed chords of the curve. Let ξ_i be the initial point of the arc whose chord is $(\Delta z)_i$. ξ_i may be chosen as any point on the arc; in the limiting process variations in sum values caused by differently chosen ξ_i's disappear. Let δ be the norm of $(\Delta z)_i$ $(i = 1, 2, \ldots n)$. If

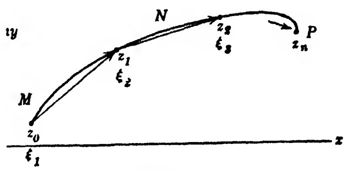


FIG. 63.

$$\lim_{\substack{\delta \to 0 \\ n \to \infty}} \sum_{i=1}^{n} f(\xi_i)(\Delta z)_i$$

exists, this limit is called the definite integral of $f(z)$ along MNP between the limits z_0 and z_n. The symbol of this limit is

$$\int_{z_0}^{z_n} f(z)dz \quad \text{or} \quad \int_{MNP} f(z)dz.$$

In the theory of functions of a complex variable only those functions of z are considered which can be expressed in the form $U(x, y) + iV(x, y)$. We shall show that line integrals in the complex plane can be made to depend upon line integrals whose integrands and differentials are real by means of the formula

$$\int_C f(z)dz = \int_C (Udx - Vdy) + i\int_C (Vdx + Udy), \qquad (311)$$

where C denotes the path of integration. Formula (311) is established directly from the definition of $\int_C f(z)dz$ as follows. Since

$$(\Delta z)_j = (\Delta x)_j + i(\Delta y)_j \qquad \begin{aligned} |(\Delta x)_j| &\leqq \delta \\ |(\Delta y)_j| &\leqq \delta \end{aligned}$$

and

$$f(\xi_j) = [U(x_j, y_j) + iV(x_j, y_j)]$$

the sum

$$S_n = \sum_1^n f(\xi_j)(\Delta z)_j = \sum_1 [U(x_j, y_j) + iV(x_j, y_j)][(\Delta x)_j + i(\Delta y)_j]$$

$$= \sum_1 [U(x_j, y_j)(\Delta x)_j - V(x_j, y_j)(\Delta y)_j]$$

$$+ i \sum_1^n [V(x_j, y_j)(\Delta x)_j + U(x_j, y_j)(\Delta y)_j].$$

Taking the limit of S_n we obtain

$$\lim_{\substack{n \to \infty \\ \delta \to 0}} S_n = \int_C f(z)dz$$

$$= \int_C (Udx - Vdy) + i\int_C (Vdx + Udy).$$

Thus $\int_C f(z)dz$ is expressed as the sum of two integrals whose integrands are real. The variable in the last two integrals may be changed by the equations $x = x(t)$ and $y = y(t)$ so that $\int_C f(z)dz$ is expressed as the integral of a single real variable t. Or we may make the transformation directly in $\int_C f(z)dz$ without using Eq. (311).

EXAMPLE 1. Evaluate the integral $\int_C zdz$, where C is the arc of the ellipse joining the points (O, b) and (a, O). Since $U = x$, $V = y$, $x = a \cos t$, and $y = b \sin t$, Eq. (311) yields

$$\int_C zdz = -\int_{\pi/2}^{0} (a^2 + b^2) \sin t \cos t dt + iab \int_{\pi/2}^{0} (\cos^2 t - \sin^2 t)dt$$

$$a^2 + b^2$$

EXAMPLE 2. Evaluate the integral $\int_C \frac{dz}{z}$, where C is the circle of radius r and center at $z = 0$. Then z (on path C) $= re^{i\theta}$, $dz = ire^{i\theta}d\theta$ and

$$\int_C \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}}\, d\theta = 2\pi i.$$

EXAMPLE 3. Evaluate the integral $\int_C (z - z_0)^n dz$, where C is the circle whose center is z_0 and whose radius is r and $n = +1, \pm 2, \pm 3, \ldots$. Let $z - z_0 = re^{i\theta}$. Then $dz = rie^{i\theta}d\theta$ and

$$\int_C (z - z_0)^n dz = \int_0^{2\pi} ir^{(n+1)}e^{i(n+1)\theta}d\theta = \frac{ir^{n+1}}{i(n+1)}\, e^{i(n+1)\theta}\Big]_0^{2\pi} = 0.$$

(To see that $e^{i(n+1)2\pi} - e^0 = 0$, it is only necessary to recall from the calculus that $e^{in\theta} = \cos n\theta + i \sin n\theta$.)

Let z_0 and z_n be two points on C, the path of integration. Then the following properties of a line integral are easily provable from either the definition of an integral or from Eq. (311).

(a) $$\int_{z_n}^{z_0} f(z)dz = -\int_{z_0}^{z_n} f(z)dz,$$

(b) $$\int_{z_0}^{z_n} [f_1(z) \pm f_2(z)]dz = \int_{z_0}^{z_n} f_1(z)dz \pm \int_{z_0}^{z_n} f_2(z)dz.$$

106. Green's Formula. The following theorem is needed in the proof of Cauchy's theorem. Let R be a simply connected region in the plane of reals bounded by a contour C. Let P and Q be any functions of x and y which together with $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are single-valued and continuous in R and on C. We shall prove:

$$\int_R\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_C (Pdx + Qdy). \tag{312}$$

This identity, Green's formula, is easily proved by the calculus. Let R be divided into subregions R_i bounded by C_i such that any *parallel to the* y*-axis cuts the contour in at most two points.* Then in R_1 (say)

$$\int_{R_1}\int \frac{\partial P}{\partial y}\, dxdy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} \frac{\partial P}{\partial y} dy = \int_a^b [P(x, y_2) - P(x, y_1)]dx. \tag{313}$$

The last integral is a line integral since y_2 and y_1 are functions of x. Since $\int_a^b P(x, y_2)dx = -\int_b^a P(x, y_2)dx$, Eq. (313) may be written

$$\int_{R_1}\int \frac{\partial P}{\partial y}\, dxdy = -\int_{C_1} Pdx \quad \text{or} \quad \int_{R_1}\int -\frac{\partial P}{\partial y}\, dxdy = \int_{C_1} Pdx$$

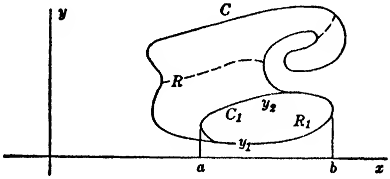


FIG. 64.

where now the line integral is taken over the closed contour C_1 in the counterclockwise direction. Similarly,

$$\int_{R_1}\int \frac{\partial Q}{\partial x}\, dydx = \int_{C_1} Qdy.$$

Formula (312) is obtained by adding the last two equations for all of the regions and contours R_i and C_i of R and C. On the common inner boundary of two regions R_i (say R_1 and R_2) the line integrals taken in opposite directions cancel.

107. Analytic Function of a Complex Variable. If U and V in Eq. (311) satisfy the conditions placed on P and Q in § 106, then Eq. (311) may be written

$$\int_C f(z)dz = -\int_R\int\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right) dxdy + i\int_R\int\left(\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}\right) dxdy.$$

Suppose U and V are subject to the additional restriction that in R and on C

$$\left.\begin{aligned} \frac{\partial U}{\partial x} &= \frac{\partial V}{\partial y}, \\ \frac{\partial U}{\partial y} &= -\frac{\partial V}{\partial x}. \end{aligned}\right\} \qquad (314)$$

Then it is evident that $\int_C f(z)dz = 0$. Eqs. (314) are known as the Cauchy-Riemann differential equations. It may seem that $f(z) = U + iV$ is greatly restricted. However, infinitely many functions $f(z)$

satisfy all these conditions, and it is in these functions that we are interested. This leads us to the definition of an analytic function. A function of a complex variable, $f(z)$, is said to be **analytic in a region R** if:

(*a*) $f(z)$ has a definite value for every z in R,

(*b*) If the expression $\dfrac{\partial U}{\partial x} + i\dfrac{\partial V}{\partial x}$ exists, has a unique value, and is continuous in R.

It will be shown in Vol. II, Chap. IV that a necessary and sufficient condition that $\dfrac{\partial U}{\partial x} + i\dfrac{\partial V}{\partial x}$ exist and have a unique value is that U_x, U_y, V_x, and V_y be continuous in R and Eqs. (314) hold for every point z of R.

EXAMPLE 1. Is $f(z) = x + iy$, an analytic function? In this case $U = x$, $V = y$ and Eqs. (314) are satisfied. Since $f(z)$ has a definite value for every value of x and y and all continuity conditions are satisfied, the function $f(z) = x + iy$ is analytic for all finite values of x and y.

The function $f(z) = x - iy$ is not analytic since Eqs. (314) are not satisfied by this function.

108. Cauchy's First Integral Theorem and Its Corollaries. Cauchy's first theorem is an obvious consequence of the definition of an analytic function and Green's formula. His theorem is: Let $f(z)$ be analytic over a connex R. Let C be any simply closed curve, the boundary of R, or lying entirely within R. Then

$$\int_C f(z)dz = 0.$$

The first corollary of Cauchy's first theorem is: Let C and C_1 be two simply closed curves such that C completely encloses C_1. Let $f(z)$ be analytic in the region between C and C_1 and on C and C_1. Then

$$\int_C f(z)dz = \int_{C_1} f(z)dz,$$

both integrations being either in the clockwise or counterclockwise direction. The positive direction of integration around a closed contour is defined to be such a direction that the area within the closed curve lies to the left of an observer on the curve and facing in the positive direction. By § 105, reversing the direction of integration

changes the sign of the result. Unless the direction is specified, positive direction is assumed.

The proof of the corollary is as follows: From Cauchy's first theorem and Fig. 65,

$$\int f(z)dz = \int_{ABD} f(z)dz + \int f(z)dz + \int f(z)dz + \int f(z)dz = 0.$$

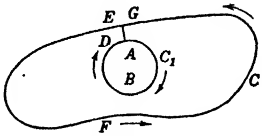


FIG. 65.

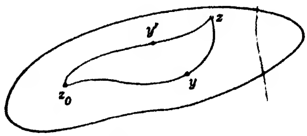


FIG. 66.

But

$$\int f(z)dz + \int f(z)dz = 0.$$

Hence

$$\int f(z)dz = -\int f(z)dz$$

or, noting directions of integration

$$\int_C f(z)dz = \int_{C_1} f(z)dz,$$

which is the corollary.

A second corollary is as follows. If $f(z)$ is analytic in a connex R, and z_0 and z are two points of R, then the value of the line integral $\int_{z_1}^{z} f(z)dz$ does not depend upon the path of integration joining z_0 and z. To see that this is true, apply Cauchy's theorem to Fig. 66, where $z_0y'z$ and z_0yz are any two different paths between z_0 and z. Since

$$\int_{z_0yzy'z_0} f(z)dz = \int_{z_0yz} f(z)dz + \int_{zy'z_0} f(z)dz = 0,$$

$$\int_{z_0yz} f(z)dz = \int_{z_0y'z} f(z)dz.$$

Thus the last integrals are independent of the path and depend only upon the end points whose coordinates are z_0 and z.

109. Definitions of Certain Elementary Functions of a Complex Variable. We seek to define some of the well-known functions of the

calculus for the case of a complex independent variable. We desire that the new definitions:

(*a*) Reduce to the definitions of the calculus of reals when the independent variable assumes real values.

(*b*) Give properties to the functions of a complex variable that as far as possible are the same as the properties of the same functions of a real variable. Conditions (*a*) and (*b*) are satisfied if e^z, $\sin z$, and $\cos z$ are defined by the equations

$$e^z = e^{x+iy} \equiv e^x(\cos y + i \sin y) \equiv 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots,$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots,$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots.$$

The function $R = \log x$ is defined in the calculus of reals as the inverse of $e^R = x$. If R and x are replaced respectively by the complex variables W and z, and $\log z$ is defined as the inverse of $z = e^W$, then W is infinitely multiple-valued. That is, to each value of z, except $z = 0$, there correspond many values of W. To see this, write z and W in the forms

$$z = r(\cos \theta + i \sin \theta),$$

$$W = U + iV.$$

Then

$$z = r(\cos \theta + i \sin \theta) = e^{U+iV} = e^U(\cos V + i \sin V).$$

Equating reals and imaginaries we obtain

$$r \cos \theta = e^U \cos V,$$

$$r \sin \theta = e^U \sin V.$$

If these equations are squared and added, we have

$$r = e^U \quad \text{or} \quad U = \log r.$$

From the same two equations $V = \theta$. If, then, we define $W = \log z$ as the inverse of $z = e^W$, we have

$$\log z = W = U + iV = \log r + i\theta = \log|z| + i \operatorname{amp} z.$$

But z may be designated in the z-plane by any of the following forms:

$$(r, \theta + 2n\pi), \text{ where } n = 0, \pm 1, \pm 2, \pm 3, \dots.$$

Thus, $\log z$ is infinitely many-valued.

The function z^a (a and z real or complex) is defined by the equation

$$z^a \equiv e^{a \log z} = e^{a \log |z|} \cdot e^{ai\theta} \cdot e^{2\pi nai}.$$

110. Integrals and Derivatives of Elementary Functions. In § 108 it was shown that the integral $\int_{z_0}^{z} f(z)dz$, where $f(z)$ is analytic, is independent of the path. By making use of this fact and Eq. (311), the value of $\int_{z_0}^{z} f(z)dz$, where $f(z)$ is an elementary function, can be found. For example, let $f(z) = e^z$ and let the path connecting z_0 and z be the straight-line segments joining $z_0 = x_0 + iy_0$ to $x + iy_0$ and $x + iy_0$ to $x + iy$. The function e^z is analytic at all finite points of the complex plane. We then have, on the path assigned, by (311) and the definition of e^z,

$$\int_{x_0+iy_0}^{x+iy} e^z dz = \int_{x_0}^{x} e^x (\cos y_0 dx - \sin y_0 dy_0) + i \int_{x_0}^{x} e^x (\sin y_0 dx - \cos y_0 dy_0)$$
$$+ \int_{y_0} e^x (\cos y dx - \sin y dy) + i \int_{y_0}^{y} e^x (\sin y dx - \cos y dy).$$

In the first integral on the right $y = y_0$ on the path of integration from $x_0 + iy_0$ to $x + iy_0$. Hence $dy = 0$. We thus have

$$\int_{x_0}^{x} e^x (\cos y_0 dx - \sin y_0 dy_0) = \int_{x_0}^{x} e^x \cos y_0 dx = \cos y_0(e^x - e^{x_0}).$$

If the other three integrals are evaluated by the same principles, the value of $\int_{z_0}^{z} e^z dz$ turns out to be

$$\int_{z_0}^{z} e^z dz = e^x(\cos y + i \sin y) - e^{x_0}(\cos y_0 + i \sin y_0) = e^z - e^{z_0}.$$

In the same way, it can be shown that:

$$\left.\begin{aligned} \int_{z_0}^{z} \sin z dz &= -(\cos z - \cos z_0), \\ \int_{z_0}^{z} \cos z dz &= \sin z - \sin z_0, \\ \int_{z_0}^{z} z^n dz &= \frac{z^{n+1}}{n+1} - \frac{z_0^{\,n+1}}{n+1} \quad (n = +1, \pm 2, \pm 3, \ldots). \end{aligned}\right\} \tag{315}$$

In the calculus of reals the derivative of $\int_{x_0}^{x} f(x)dx$ with respect to x $f(x)$. Similarly, in the theory of functions the derivative is defined as the inverse process to that of integration. We thus have from Eqs. (315) the relations

$$\frac{d \cos z}{dz} = -\sin z,$$

$$\frac{d \sin z}{dz} = \cos z,$$

$$\frac{d}{dz}\frac{z^{n+1}}{n+1} = z^n.$$

111. Taylor's Series and Singular Points. It is recalled from the calculus that if $f(x)$ and its first n derivatives exist and are continuous over the interval $a \leqq x \leqq b$, then $f(x)$ may be expanded in that interval in a polynomial of the form

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \ldots + f^{(n)}(x_1)\frac{(x-a)^n}{n!}, \quad (316)$$

where $f', f'', \ldots f^{(n)}$ are respectively the first, second, and nth derivatives of $f(x)$ and $a < x_1 < b$. Eq. (316) is Taylor's formula. As n approaches infinity in (316), Taylor's formula becomes Taylor's infinite series. Taylor's infinite series represents $f(x)$ for those values of x, and those only, for which $\frac{f^{(n)}(x_1)(x-a)^n}{n!}$ approaches zero as n approaches infinity. Taylor's expansion is unique if it exists. It is also remembered from the calculus that $f(x) = \log x$ cannot be expanded in a power series in $(x - a)$ where $a = 0$. The point $x = 0$ is a singularity of $\log x$.

It is proved in Vol. II, Chap. IV, that Taylor's series holds for analytic functions of a complex variable. It is

$$f(z) = f(a) + f'(a)(z - a) + \frac{f''(a)}{2!}(z - a)^2 + \ldots.$$

When a function is said to be analytic this does not mean that it is analytic for every point of the z-plane or for values of z which are infinite. In fact, there is a theorem which states that the only functions analytic for all values of z are constants. A function is called analytic if it is analytic in some region. Thus for all functions (constants excepted), there exist values of z, finite or infinite, for which

$f(z)$ is not analytic. For these values of z, the function does not satisfy the conditions stated in § 107. Moreover, it will be shown later that there exists no Taylor's expansion at such points, i.e., if $f(z)$ is not analytic at a then Taylor's series is not valid. If $f(z)$ can be expanded in a Taylor's series which converges for all points about a and interior to a circle whose radius is greater than zero then $f(z)$ is called a regular function at a, and a is called a **regular point** of $f(z)$. Points in the z-plane which are not regular points of $f(z)$ are **singular points** of $f(z)$.

112. Singular Points. Poles, points of discontinuity, and essential singularities are singular points of single-valued complex functions; these same singularities and in addition branch points are singular points of multiple-valued functions. If, in the expansion of $f(z)$ in a Taylor's series,

$$f(a) = f'(a) = \ldots f^{(n-1)}(a) = 0$$

and

$$f^{(n)} \neq 0,$$

then

$$f(z) \quad \frac{f^{(n)}(a)}{n!}(z - a)^n + \frac{f^{(n+1)}(a)}{(n+1)!}(z - a)^{n+1} + \ldots,$$

and the point a (real or complex) is said to be a zero of order n of $f(z)$. The function $f(z)$ in this case may be written

$$f(z) = (z - a)^n g(z),$$

where $g(a) \neq 0$. If a is a zero of order n of $\frac{1}{f(z)}$, then a is a pole of order n of $f(z)$, and conversely. Or the definition of a pole may be stated in the following way. If $f(z)$ may be written in the form

$$f(z) = \frac{g(z)}{(z - a)^n},$$

where $g(z)$ is analytic at $z = a$ and $g(a) \neq 0$, then a is a pole of $f(z)$ of order n. Essential singularities will be examined in § 113 and branch points, when needed, in the next section of this chapter.

Example. By inspection, determine the singularities of $f(z) = \frac{\text{[illegible]}}{z^2 + 1}$. Since $\frac{\text{[illegible]}}{z^2 + 1} = \frac{\text{[illegible]}}{(z + i)(z - i)}$, $f(z)$ has simple poles, that is, poles of the first order, at $z = i$ and $z = -i$. The expression e^z becomes infinite at $z = \infty$. The function $f(z)$ is continuous except at $z = \pm i$ and ∞. If $f(z)$ has a branch point then, by the definition

of a branch point given later, it is possible to obtain a value of z for which $f(z)$ is at least double-valued. This is not possible for $f(z) = \frac{e^z}{z^2 + 1}$. Consequently the singularities of $f(z)$ are poles at $z = \pm i$ and an essential singularity at $z = \infty$.

113. Laurent's Expansion. Suppose that the point a is a regular point or a pole or an isolated essential singularity of $f(z)$, but that $f(z)$ is analytic within the annular connex R between two concentric circles whose center is a and whose radii are r_1 and r_2. Then it can be proved (See Vol. II, Chap IV.) that within the annular region R

$$f(z) = \sum_{n=-\infty}^{n=+\infty} C_n(z - a)^n, \tag{317}$$

where the constants C_n are given by the formula

$$C_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z - a)^{n+1}} \quad (n = 0, \pm 1, \pm 2, \pm 3, \ldots)$$

and where C is a circle whose radius is r, $(r_1 < r < r_2)$ and whose center is a. The series in Eq. (317) is called **Laurent's expansion** of $f(z)$. One of the uses of this expansion is the investigation of the nature of the singular points of single-valued analytic functions.

It is provable that this expansion, like Taylor's expansion, is unique, i.e., there is only one two-way series in positive and negative powers of $(z - a)$ with constant coefficients which represents $f(z)$ in the region R. Consequently, if such a two-way series can be found by any method it is Laurent's expansion for the given function $f(z)$. Because the C_n are, in general, difficult to evaluate by the above formula, a Laurent series is usually obtained by the easy method of the illustrative example of this article.

If a is a regular point of $f(z)$, then $C_n = 0$ for $n = -1, -2, -3, \ldots$. If a is a pole of order m then (317), as will appear from the illustrative example following, is of the form

$$f(z) = \sum_{n=-m}^{n=\infty} C_n(z - a)^n.$$

But if the point a is an isolated essential singularity then the series (317), the two-way infinite series, holds where the C_n for $n = -1, -2, -3, \ldots -\infty$ are not zero. This is the distinguishing characteristic (and a definition) of an essential singularity. For example,

consider the function $e^{1/z}$. From

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots = \sum_0^\infty \frac{z^n}{n!}$$

we obtain

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots = \sum_0^\infty \frac{1}{n!z^n}$$

$$= \sum_{-\infty}^{0} \frac{z^n}{(-n)!} = \sum_{-\infty}^{0} \frac{(z-0)^n}{(-n)!}$$

The above series is the Laurent expansion for $e^{1/z}$ about the point $a = 0$, and because it contains powers of $(z - 0)$ infinitely large and negative, $e^{1/z}$ has an essential singularity at the point $z = 0$.

EXAMPLE. It is desired to expand $f(z) = \dfrac{\sin z}{z^2(z+2)}$ in a Laurent series about the point $z = 0$, and to determine the nature of the singularity there. We may write

$$f(z) = \frac{\sin z}{2z^2\left(1 + \frac{z}{2}\right)} = \frac{1}{2}(\sin z)z^{-2}\left(1 + \frac{z}{2}\right)^{-1},$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \quad \text{for all finite values of } z,$$

$$\left(1 + \frac{z}{2}\right)^{-1} = 1 - \frac{z}{2} + \frac{z^2}{4} - \cdots \quad \text{for } |z| < 2.$$

The last expansion is obtained by the binomial theorem. We may multiply these series, provided $z < 2$. Finally then,

$$f(z) = \frac{\sin z}{2z^2\left(1 + \frac{z}{2}\right)} = \frac{1}{2z} - \frac{1}{4} + \frac{z}{24} - \cdots.$$

The function $f(z)$ thus has a pole of the first order at $z = 0$.

114. Residues and the Residue Theorem. Let a be a pole of order m, or an isolated singular point, of $f(z)$. Then the residue of $f(z)$ at a is defined to be the coefficient, C_{-1}, of $\dfrac{1}{z-a}$ in the Laurent expansion. But

$$C_{-1} = \frac{1}{2\pi i}\int_C f(z)dz.$$

Hence an equivalent definition of a residue is the line integral,

$$R = \frac{1}{2\pi i}\int_C f(z)dz,$$

where R denotes the residue, and C encloses no poles except a.

Let $f(z)$ have a finite number of poles or isolated singularities a_n ($n = 1, 2, 3, \ldots m$) in a connex R which is bounded by a closed curve G. For simplicity, consider m to be 3; then we have the three singularities a_1, a_2, and a_3. Since $f(z)$ is analytic except at a_1, a_2, and a_3, we have from Cauchy's theorem

$$\int_G f(z)dz + \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz = 0,$$

where C_1, C_2, and C_3 are small circles enclosing a_1, a_2, and a_3, respectively, and if the directions of integration are as shown in Fig. 67.

The complete path is shown in the figure. The net integral along each cut vanishes as the two sides of the cut approach coincidence. The positive direction of integration, however, is defined as such a direction that the area enclosed by the contour lies to the left of an observer on the contour and facing in the positive direction of integration. Hence, as written before, the integrals about C_1, C_2, C_3 were taken in the negative direction. If we write the integrals as taken in the positive direction,

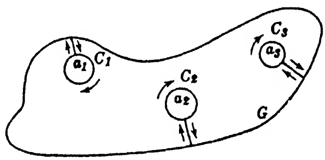


FIG. 67.

$$\int_G f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz.$$

But, by the second definition of a residue,

$$\int_{C_1} f(z)dz = 2\pi i R_1,$$

where R_1 is the residue of $f(z)$ at a_1. Likewise, in general,

$$\int_{C_n} f(z)dz = 2\pi i R_n \quad (n = 1, 2, 3, \ldots, m)$$

and

$$\int_G f(z)dz = 2\pi i \sum_{n=1}^{m} R_n,$$

which is the residue therem. One of the chief uses of the residue theorem is the evaluation of integrals in the complex plane.

EXAMPLE. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}.$$

This particular integral is more easily evaluated by integration along the x-axis, but we evaluate it here by integration in the plane in order to illustrate the use of the residue theorem. Although this is an integral involving a real variable only, it can be transformed into a contour integral in the complex plane. Evidently the given integral may be replaced by one in the complex variable z, provided the limits remain the same, and the path be the x-axis. We consider the related contour (or line) integral.

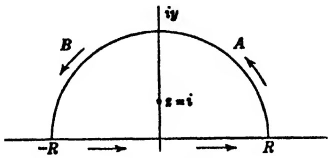


FIG. 68.

$$\int_{-RRAB(-R)} \frac{dz}{(z^2+1)^3},$$

where the path is the semicircle $-RRAB(-R)$ whose diameter is $-RR$. (See Fig. 68.)

By the residue theorem

$$\int_{-RRAB(-R)} \frac{dz}{(z^2+1)^3} = 2\pi i \sum^{m} R_n,$$

where the number of residues is m, one residue for each pole. The poles occur at the zeros of the denominator, at

$$z^2 + 1 = 0 \quad \text{or} \quad z^2 = -1, \quad z = \pm i.$$

The path in the above figure includes but one pole, namely $z = i$. Let us determine the residue at this pole from the Laurent expansion of the integrand about $z = i$; i.e., we expand the integrand as a power series in $(z - i)$. To do this, write

$$\frac{1}{(z^2+1)^3} = \frac{1}{(z-i)^3}\frac{1}{(z+i)^3}$$

and expand $\dfrac{1}{(z+i)^3}$ in a Taylor's series of the form

$$f(z) = f(i) + f'(i)(z-i) + \frac{f''(i)(z-i)^2}{2!} + \cdots.$$

By this formula

$$(z+i)^{-3} = (2i)^{-3} + (-3)(2i)^{-4}(z-i) + \frac{(3\cdot 4)(2i)^{-5}(z-i)^2}{2!} + \cdots,$$

$$= +\frac{i}{8} - \frac{3}{16}(z-i) - \frac{3i}{16}(z-i)^2 + \cdots.$$

Hence

$$\frac{1}{(z^2+1)^3} = \frac{(z+i)^{-3}}{(z-i)^3} = \frac{+i}{8(z-i)^3} - \frac{3}{16(z-i)^2} - \frac{3i}{16(z-i)} + \cdots.$$

Thus the residue is $-\dfrac{3i}{16}$, and

$$\int_{-RRAB(-R)} \frac{dz}{(z^2+1)^3} = 2\pi i\left(-\frac{3i}{16}\right) = \frac{3\pi}{8}.$$

But also, by reference to Fig. 68,

$$\int_{-RRAB(-R)} \frac{dz}{(z^2+1)^3} = \int_{-R}^{R} \frac{dz}{(z^2+1)^3} + \int_{RAB(-R)} \frac{dz}{(z^2+1)^3}$$

Now let the radius of the semicircle R approach ∞. As $R \to \infty$ the integral $\displaystyle\int \frac{dz}{(z^2+1)^3} \to 0$, since if z be set equal to $Re^{i\theta}$ we have

$$\int_{RAB(-R)} \frac{dz}{(z^2+1)^3} = \frac{i}{R^5}\int_0^{\pi} \frac{e^{i\theta}d\theta}{\left(e^{2i\theta} + \dfrac{1}{R^2}\right)^3}$$

which surely approaches zero as $R \to \infty$. Finally,

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8}.$$

EXERCISES

1. By reference to the Laurent expansion, prove the residue rules:

(a) If a is a pole of order r of $f(z)$, the residue of $f(z)$ at $z = a$ is the coefficient of $(z-a)^{r-1}$ in the development of the product $(z-a)^r f(z)$.

(b) If a is a simple pole of $f(z)$, the residue of $f(z)$ at $z = a$ is equal to

$$\lim_{z\to a} (z-a)\, f(z).$$

(*c*) If $r(z) \; \frac{P(z)}{Q(z)}$ where both $P(z)$ and $Q(z)$ are regular at $z - a$, and $P(a) = 0$ and $f(z)$ has a simple pole at $z = a$, then the residue of $f(z)$ is equal to $\frac{P(a)}{Q'(a)}$ where Q' denotes the derivative.

2. Evaluate: $\int_{-\infty}^{\infty} \frac{(x^2 - x + 2)}{(x^2 + 1)(x^2 + 9)} dx.$

3. Considering $\int \frac{e^{azi}}{z^3 + 1} dz$ around the contour show that

$$\int_{-\infty}^{\infty} \frac{\cos ax}{1 + x^3} dx = \frac{\pi}{3} \sin a + \frac{2\pi}{3} e^{-\frac{1}{2}a\sqrt{3}} \cos\left(\frac{a}{2} - \frac{\pi}{6}\right.$$

$$\int_{-\infty}^{\infty} \frac{\sin ax}{1 + x^3} dx = \frac{\pi}{3} \cos a + \frac{2\pi}{3} e^{-\frac{1}{2}a\sqrt{3}} \sin\left(\frac{a}{2} \quad 6\right),$$

where $a > 0$.

(On the small semicircle let $z + 1 = re^{i\theta}$, and on the large semicircle $z = Re^{i\theta}$. Then let $r \to 0$ and $R \to \infty$.)

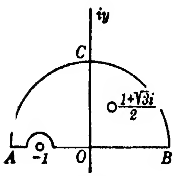


FIG. 68*a*.

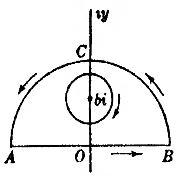


FIG. 68*b*.

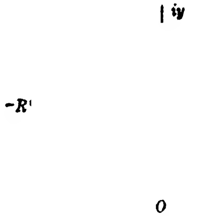


FIG. 68*c*.

4. Expand $\frac{\cdot}{1 + z^2}$ in a Laurent expansion about the pole $z = i$.

5. Show that $\int_{-\infty}^{\infty} \frac{e^{iax}}{b^2 + x^2} dx = \frac{\pi e^{-ab}}{b}$, ($b$ real and > 0) ($1 >$ a real > 0).

Hint: Use contour shown.

6. From Ex. 5 prove that

$$\int_{-\infty}^{\infty} \frac{\cos axdx}{b^2 + x^2} = \frac{\pi e^{-ab}}{b},$$

$$\int_{-\infty}^{\infty} \frac{\sin axdx}{b^2 + x^2} = 0.$$

7. Show that $\int_{-\infty}^{\infty} e^{-x^2} \cos 2\, bxdx = \sqrt{\pi} e^{-b^2}.$

Hint: Use the contour shown and the relation $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$

II

RIGOROUS AND SYSTEMATIC PROOFS OF HEAVISIDE'S RESULTS

Heaviside's operational results consist mainly of the three well-known Heaviside rules. We are now in a position to establish rigorously these rules.

115. Heaviside's Circuit Problems. Heaviside's circuit problems may be grouped under two headings: (*a*) "Unextended Problems," and (*b*) "Extended Problems." His unextended problem is as follows: Given a linear network of n meshes in a state of equilibrium (no currents flowing or charges existing); find its response when a "unit" electromotive force is applied in any mesh. His "unit" function. usually written $\mathbf{1}$ or $\mathit{1}$, is defined to be a voltage which is zero for $t < 0$ and unity for $t \geq 0$.

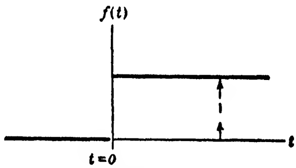


FIG. 69.

The mathematical statement of Heaviside's unextended problem is: *Given the* n *simultaneous linear differential equations with constant coefficients which specify the performance of an* n*-mesh network; find the response of this network when a unit electromotive force appears in one mesh. That is, given*

$$\begin{aligned} z_{11}i_1 + z_{12}i_2 + \ldots + z_{1n}i_n &= \mathit{1}, \\ z_{21}i_1 + z_{22}i_2 + \ldots + z_{2n}i_n &= 0, \\ & \\ z_{n1}i_1 + z_{n2}i_2 + \ldots + z_{nn}i_n &= 0, \end{aligned} \tag{318}$$

find the solution subject to the condition that the network was initially in a state of equilibrium. In these equations

$$z_{rs} = L_{rs}p + R_{rs} + \frac{1}{C_{rs}p}$$

is the **generalized impedance** of the rth mesh to the sth current, i_s; L_{rs}, R_{rs}, and C_{rs} are constants, and

$$pi_s = \frac{di_s}{dt}, \; \frac{1}{p}i_s = \int_0^t i_s dt = q_s,$$

where q_s is the charge that has flowed in the sth mesh. Methods of writing circuit Eqs. (318) for a given network are found in § 20.

Eqs. (318) are adequate for all circuits having lumped parameters L, R, and C for which Kirchhoff's laws are adequate. (See § 20.)

The current produced in mesh 1 due to the unit e.m.f. being applied in that mesh is called the **indicial admittance,** and is denoted by $A(t)$. The current i_k produced in the kth mesh due to the unit e.m.f. being applied in the jth mesh (j different from k) is called the transfer indicial admittance, and is denoted by $A_{jk}(t)$.

The "extended Heaviside problem" differs from the unextended in that there is at least one applied electromotive force which is a variable function of the time. Stated mathematically, given the differential equations

$$\begin{aligned} z_{11}i_1 + z_{12}i_2 + \ldots + z_{1n}i_n &= 1 f_1(t), \\ z_{21}i_1 + z_{22}i_2 + \ldots + z_{2n}i_n &= 1 f_2(t), \\ & \\ z_{n1}i_1 + z_{n2}i_2 + \ldots + z_{nn}i_n &= 1 f_n(t), \end{aligned} \qquad (319)$$

where at least one $f(t)$ is neither zero nor the unit function, find the solution subject to the condition that the network was initially in a state of equilibrium.

Heaviside obtained three rules for the solution of the unextended problem which we will give later. He was not particularly concerned with the proof of these rules, but was primarily interested in their application. (The proof of § 116 although original was suggested by Bromwich's proof, Ref. 39.)

116. The Solution of Heaviside's Unextended Problem. The solution of the system of differential equations (318) satisfying the initial conditions of equilibrium

$$\begin{aligned} i_1(0) &= i_2(0) = \ldots = i_n(0) = 0, \\ q_1(0) &= q_2(0) = \ldots = q_n(0) = 0, \end{aligned} \qquad (320)$$

is

$$i_k = \Phi(p)1 = \frac{1}{2\pi i}\int_C \frac{\Phi(\lambda)e^{\lambda t}d\lambda}{\lambda} \; (k = 1, 2, 3, \ldots, n), \qquad (321)$$

where

$$\Phi(p) = \frac{M_{1k}(p)}{D(p)}$$

and C is either a closed curve enclosing all the roots of $\lambda D(\lambda) = 0$, or a line from $-i\infty$ to $+i\infty$ of such form that all singularities of the integrand be on the left. $D(p)$ is the determinant of the coefficients

of $i_1\, i_2, \ldots, i_n$ in the system (318), and $M_{1k}(p)$ is the cofactor of Z_{1k} in $D(p)$. (For definition of cofactor, see § 28.)

We will now obtain the solution (321). A relation of the form

$$i_k = \frac{1}{2\pi i}\int \xi_k(\lambda)e^{\lambda t}d\lambda \; (k = 1, 2, \ldots, n) \; (t \geqq 0) \tag{322}$$

is desired which satisfies (318) and (320). If (322) for $k = 1$ is substituted in the first term of the first equation of (318) we have in detail

$$\begin{aligned} z_{11}(p)i_1 &= \left(L_{11}p + R_{11} + \frac{1}{C_{11}p}\right)\frac{1}{2\pi i}\int_C \xi_1(\lambda)e^{\lambda t}d\lambda \\ &= \frac{1}{2\pi i}\int_C \left[L_{11}\lambda e^{\lambda t} + R_{11}e^{\lambda t} + \frac{1}{C_{11}\lambda}(e^{\lambda t} - 1)\right]\xi_1(\lambda)d\lambda \\ &= \frac{1}{2\pi i}\int_C \lambda_{11}\xi_1(\lambda)e^{\lambda t}d\lambda - \frac{1}{2\pi i}\int_C \frac{\xi_1(\lambda)}{C_{11}\lambda}d\lambda, \end{aligned}$$

where $\lambda_{11} = z_{11}(\lambda)$. Similarly, the substitution of (322) throughout Eqs. (318) gives

$$\left.\begin{aligned} &\int_C [\lambda_{11}\xi_1(\lambda) + \ldots + \lambda_{1n}\xi_n(\lambda)]e^{\lambda t}d\lambda \\ &\qquad - \int_C \left[\frac{\xi_1(\lambda)}{C_{11}\lambda} + \ldots + \frac{\xi_n(\lambda)}{C_{1n}\lambda}\right]d\lambda = 2\pi i\, 1 \\ &\int_C \Big[[\lambda_{21}\xi_1(\lambda) + \ldots + \lambda_{2n}\xi_n(\lambda)]e^{\lambda t}d\lambda \\ &\qquad - \int_C \left[\frac{\xi_1(\lambda)}{C_{21}\lambda} + \ldots + \frac{\xi_n(\lambda)}{C_{2n}\lambda}\right]d\lambda = 0 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\int_C [\lambda_{n1}\xi_1(\lambda) + \ldots +)\lambda_{nn}\xi_n(\lambda)]e^{\lambda t}d\lambda \\ &\qquad - \int_C \left[\frac{\xi_1(\lambda)}{C_{n1}\lambda} + \ldots + \frac{\xi_n(\lambda)}{C_{nn}\lambda}\right]d\lambda = 0, \end{aligned}\right\} \tag{323}$$

where

$$\lambda_{rs} = Z_{rs}(\lambda)\; (r, s = 1, 2, \ldots, n).$$

Since

$$\int_C \frac{e^{\lambda t}}{\lambda}d\lambda = \int_C \frac{1 + \lambda t + \frac{(\lambda t)^2}{2!} + \ldots}{\lambda}d\lambda = \int_C \frac{d\lambda}{\lambda} = 2\pi i$$

(See examples 2 and 3, § 105), Eqs. (323) are evidently satisfied if

$$\lambda_{11}\xi_1(\lambda) + \ldots + \lambda_{1n}\xi_n(\lambda) - \left[\frac{\xi_1(\lambda)}{C_{11}\lambda} + \ldots + \frac{\xi_n(\lambda)}{C_{1n}\lambda}\right]e^{-\lambda t} = \frac{1}{\lambda}1,$$

$$\lambda_{21}\xi_1(\lambda) + \ldots + \lambda_{2n}\xi_n(\lambda) - \left[\frac{\xi_1(\lambda)}{C_{21}\lambda} + \ldots + \frac{\xi_n(\lambda)}{C_{2n}\lambda}\right]e^{-\lambda t} = 0,$$

$$\lambda_{n1}\xi_1(\lambda) + \ldots + \lambda_{nn}\xi_n(\lambda) - \left[\frac{\xi_1(\lambda)}{C_{n1}\lambda} + \ldots + \frac{\xi_n(\lambda)}{C_{nn}\lambda}\right]e^{-\lambda t} = 0.$$

Although the last equations have no useful solution in the $\xi_k(\lambda)$ yet they suggest a solution of Eqs. (323). Neglecting the coefficients of $e^{-\lambda t}$ and recalling that for $t \geqq 0$ the unit function is the real number one we replace tentatively the last equations by Eqs. (324).

$$\begin{aligned}\lambda_{11}\xi_1(\lambda) + \lambda_{12}\xi_2(\lambda) + \ldots + \lambda_{1n}\xi_n(\lambda) &= 1\\ \lambda_{21}\xi_1(\lambda) + \lambda_{22}\xi_2(\lambda) + \ldots + \lambda_{2n}\xi_n(\lambda) &= 0,\\ \\ \lambda_{n1}\xi_1(\lambda) + \lambda_{n2}\xi_2(\lambda) + \ldots + \lambda_{nn}\xi_n(\lambda) &= 0,\end{aligned} \tag{324}$$

If (324) are solved for ξ_k the value so obtained, owing to properties of line integrals, also satisfies Eqs. (323). If the determinant

$$D(\lambda) = \begin{vmatrix} \lambda_{11} & \lambda_{12} \ldots \lambda_{1n} \\ \lambda_{21} & \lambda_{22} \ldots \lambda_{2n} \\ \cdot & \cdot \\ \lambda_{n1} & \lambda_{n2} \ldots \lambda \end{vmatrix}$$

is not zero, the solution of (324) is

$$\xi_k(\lambda) = \frac{M_{1k}(\lambda)}{\lambda D(\lambda)}, \tag{325}$$

where $M_{1k}(\lambda)$ is the cofactor of λ_{1k} in $D(\lambda)$. The substitution of (325) in (322) gives (321).

It is now necessary to show that the solution (325) reduces the value of the second line integral in each equation of the system (323) to zero. Substitute (325) into the second integral of the qth equation of (323) to obtain

$$\int_C \left[\frac{M_{11}(\lambda)}{C_{q1}} + \frac{M_{12}(\lambda)}{C_{q2}} + \ldots + \frac{M_{1n}(\lambda)}{C_{qn}}\right]\frac{d\lambda}{\lambda^2 D(\lambda)} (q = 1, 2, \ldots, n). \tag{326}$$

A typical term of the above integral may be written

$$\int_C \frac{M_{1m}(\lambda)}{C_{qm}\lambda^2 D(\lambda)}\,d\lambda = \int_C \frac{\lambda^n M_{1m}(\lambda)}{C_{qm}\lambda^{n+2} D(\lambda)}\,d\lambda.$$

Now $\lambda \cdot \lambda_{r_s}$ is a quadratic function of λ, whence $\lambda^n M_{1k}$ is a polynomial of degree $2n - 1$ in λ, and $\lambda^{n+2}D(\lambda)$ is a polynomial of degree $2n + 2$ in λ. Hence, the integrand of the integral may be written

$$\frac{1}{\lambda^3}\left[\frac{a_0\lambda^{2n-1} + a_1\lambda^{2n-2} + \ldots + a_{2n-1}}{b_0\lambda^{2n-1} + b_1\lambda^{2n-2} + \ldots + b_{2n-1}}\right]$$

$$= \frac{1}{\lambda^3}\left[\frac{a_0 + \dfrac{a_1}{\lambda} + \ldots + \dfrac{a_{2n-1}}{\lambda^{2n-1}}}{b_0 + \dfrac{b_1}{\lambda} + \ldots + \dfrac{b_{2n-1}}{\lambda^{2n-1}}}\right].$$

The last fraction may be expanded in a Maclaurin series in powers of $1/\lambda$, and the integrand may be written

$$\frac{\lambda^n M_{1k}(\lambda)}{\lambda^{n+1}D(\lambda)} = \frac{1}{\lambda^3}\left(C_0 + \frac{C_1}{\lambda} + \frac{C_2}{\lambda^2} + \ldots\right).$$

The above series converges provided λ is sufficiently large. Since the path of integration C encloses all the singularities, we may take it to be a circle of such great radius that this series converges at all points on it. By § 105, example 3,

$$\int_C \frac{d\lambda}{\lambda^n} = 0. \quad (n \neq 1.)$$

Ccnsequently, the integral (326) is zero, and it follows that (325) is the solution of (323).

It remains to show that the initial conditions are satisfied. From (321), when $t = 0$,

$$i_k(0) = \frac{1}{2\pi i}\int_C \frac{M_{1k}(\lambda)d\lambda}{\lambda D(\lambda)},$$

and

$$q_k(0) = \frac{1}{2\pi i}\int_C \frac{M_{1k}(\lambda)d\lambda}{\lambda^2 D(\lambda)}.$$

Both these integrals may be shown to be zero by the method in reducing (326), hence the initial conditions are satisfied, and the proof is complete.

117. Operational Formulas. If Eqs. (318) are solved for $i_k(k = 1, 2, \ldots, n)$, treating p as a mere algebraic symbol, we obtai..

$$i_k = \frac{M_{1k}(p)}{D(p)} 1 = \Phi(p) 1.$$

Now it is known from the theory of differential equations that (318) has, subject to the boundary conditions (320), precisely one solution. This solution is given by Bromwich's integral

$$i_k = \frac{1}{2\pi i}\int_C \frac{M_{1k}(\lambda)e^{\lambda t}}{\lambda D(\lambda)}\, d\lambda.$$

The integrand of this integral can, of course, be transformed in many ways: resolved into partial fractions, expanded in series, etc. Consequently, the operator expression $\Phi(p) 1 = \frac{M_{1k}(p)}{D(p)} 1$ may be similarly transformed as if p were merely an algebraic symbol. This is a result of great importance. There exist infinitely many algebraic operational formulas $\Phi(p) 1$ which can be evaluated by Bromwich's integral. Two of these are given in the illustrative examples of this article, and many more are given in the exercises of this chapter.

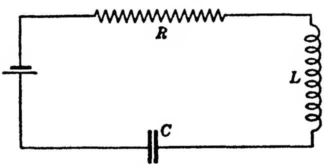


FIG. 70.

EXAMPLE 1. Find the response of the simple series circuit, Fig. 70, when unit voltage is applied, the previous current and condenser charge each being zero. Take $R = 12$ ohms, $L = 1$ henry, $C = 0.01$ farad The differential equation is found by the method of § 20 to be

$$\left(Lp + R + \frac{1}{pC}\right)i_1 = 1,$$

or

$$\frac{(1.0p^2 + 12.0p + 100)}{p} i_1 = 1.$$

Whence

$$i_1 = \frac{p}{p^2 + 12p + 100} 1 = \Phi(p) 1.$$

Then from (321),

$$i_1 = \frac{1}{2\pi i}\int_C \frac{e^{\lambda t} d\lambda}{\lambda^2 + 12\lambda + 100}.$$

The integrand has poles at $\lambda = -6 + i8$ and $\lambda = -6 - i8$, whence by the residue theorem and Ex. 1*b* of § 114.

$$
\begin{aligned}
i_1 &= \left.\frac{e^{\lambda t}}{\lambda + 6 - i8}\right|_{\lambda=-6-i8} + \left.\frac{e^{\lambda t}}{\lambda + 6 + i8}\right|_{\lambda=-6+i8} \\
&= \frac{e^{(-6-i8)t}}{-i16} + \frac{e^{(-6+i8)t}}{i16} \\
&= \frac{e^{-6t}}{8}\left(\frac{e^{i8t} - e^{-i8t}}{2i}\right) = \frac{1}{8}e^{-6t}\sin 8t.
\end{aligned}
$$

and the current is a damped sinusoid.

Example 2. The evaluation of

$$
i_1 = \frac{p^2}{p^2 + \omega^2}\,1
$$

may be accomplished by means of (321) as follows:

$$
\begin{aligned}
i_1 &= \frac{1}{2\pi i}\int_C \frac{\lambda^2 e^{\lambda t}}{\lambda(\lambda^2 + \omega^2)}\,d\lambda \\
&= \frac{1}{2\pi i}\int_C \frac{\lambda e^{\lambda t}}{(\lambda + i\omega)(\lambda - i\omega)}\,d\lambda.
\end{aligned}
$$

The poles occur at $\lambda = \pm i\omega$, whence the integral is

$$
\begin{aligned}
i_1 &= \left.\frac{\lambda e^{\lambda t}}{\lambda + i\omega}\right|_{\lambda=i\omega} + \left.\frac{\lambda e^{\lambda t}}{\lambda - i\omega}\right|_{\lambda=-i\omega} \\
&= \frac{e^{i\omega t}}{2} + \frac{e^{-i\omega t}}{2} = \cos \omega t.
\end{aligned}
$$

118. Heaviside's Expansion Theorem. Heaviside's first rule or expansion theorem gives the indicial admittance for a network of n meshes in the form

$$
A(t) = \frac{M_{11}(0)}{D(0)} + \sum_{k=1}^{2n} \frac{M_{11}(\lambda_k)e^{\lambda_k t}}{b\lambda_k(\lambda_k - \lambda_1)(\lambda_k - \lambda_2)\ldots}, \tag{327}
$$

where $\lambda_1, \lambda_2, \ldots \lambda_{2n}$ are the roots of $D(\lambda) = 0$ (all λ's being distinct), and b is the coefficient of λ^{2n} in $D(\lambda)$, i.e., $D(\lambda) = b(\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_{2n})$. The denominator of the summation contains all the factors $\lambda_k - \lambda$, which are not zero.

The proof of (327) is accomplished from the Bromwich integral (321) by inspection. That is, if the residues of the line integral are calculated and added, formula (327) results. Since the process of calculating residues at simple poles is easy to remember whereas formula (327) is remembered with difficulty, the expansion theorem is seldom used. It is given, in this treatment, only for completeness.

If $D(\lambda) = 0$ has zero or multiple roots, no difficulty arises. It is necessary only to evaluate the residues at multiple poles as indicated in Ex. 1 at the end of the first section of this chapter. The roots of $D(\lambda) = 0$ are found by Graeffe's method (§ 46).

Heaviside's second rule is: If $\Phi(p)$ can be expanded in a convergent series

$$\Phi(p) = a_0 + \frac{a_1}{p} + \frac{a_2}{p^2} + \frac{a_3}{p^3} + \dots,$$

then

$$\Phi(p)\,1 = a_0 + a_1 t + \frac{a_2 t^2}{2!} + \frac{a_3 t^3}{3!} + \dots. \tag{328}$$

The proof of this rule follows at once from the Bromwich integral. If the indicial admittance is desired for small values of the time (328) may be satisfactory. However, in general, (328) is used only as a last resort because the properties of the solution are frequently lost when expressed in series form.

119. Fractional Powers of p. In the application of Heaviside's third rule, § 120, to transmission lines and to heat-flow problems, fractional powers of p operating on unit function are encountered. But to evaluate $p^n 1 (0 < n < 1)$ by the Bromwich integral it is first necessary to understand branch points of multiple-valued analytic functions.

Consider the function $W^2 = z$, which may be written

$$W = z^{\frac{1}{2}} = +\sqrt{r}\left(\cos\frac{\theta + 2\pi k}{2} + i\sin\frac{\theta + 2\pi k}{2}\right),$$

where $k = 0$ or 1. It is possible to consider $\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$ and $\sqrt{r}\left(\cos\frac{\theta + 2\pi}{2} + i\sin\frac{\theta + 2\pi}{2}\right)$ as two distinct functions. Such is not done, but they are considered as two branches W_1 and W_2 of the one function W. We are thus led to define a branch of a function. Let there be a set of pair values (W, z), such that all the z's considered fill exactly once a region R. Let there be a one-to-one correspondence

between the points of the region R and a region in the W-plane. The points in the W-plane defined by the one-to-one correspondence form a continuous function of z and are called a **branch** of the function W. In the example $W^2 = z$, let z traverse in the z-plane any path $ABCDE$ which encircles the origin. (This path is the region occupied by z.)

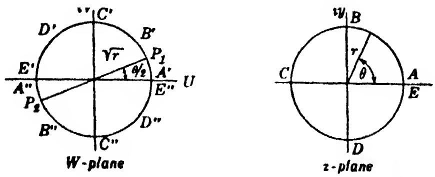


FIG. 71.

For simplicity, let the path be a circle whose radius is r (Fig. 71). Then $z = re^{i\theta}$. As θ varies from 0 to 2π, $\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = \sqrt{r}\, e^{i(\theta/2)}$ describes a semicircle $A'B'C'D'E'$ of radius $\sqrt{r}$, in the upper half of the W-plane, while $\sqrt{r}\left(\cos\frac{\theta + 2\pi}{2} + i\sin\frac{\theta + 2\pi}{2}\right) = \sqrt{r}\, e^{i(\theta+2\pi)/2}$ describes a semicircle $A''B''C''D''E''$ of the same radius in the lower half of the W-plane. When θ (in the z-plane) takes on the value 2π, the branches of W interchange. That is, if a point P_1 traversed the branch $A'C'E'$ while the point P_2 traversed the branch $A''C''E''$ as θ increased from θ to 2π, then as θ increases from 2π to 4π, P_2 traverses $A'C'E'$, and P_1 traverses $A''C''E''$.

In general, a point a is a **branch point** of the function $W = f(z)$ if the branches of W interchange as z encirles a. In the example

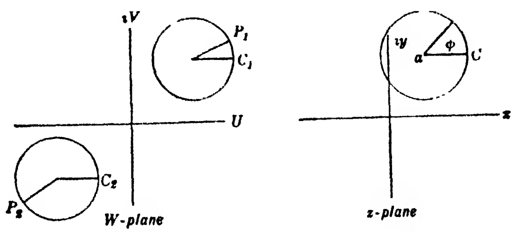


FIG. 72.

$W^2 = z$, the branch point is $z = 0$. Suppose z traverses a curve C in the z-plane which does not encircle the origin. (See Fig. 72.) Then, as ϕ varies from 0 to 2π, W_1 and W_2 describe the two closed curves C_1 and C_2. As ϕ increases from 2π to 4π, each of the branches is

described again, but the point P_1 on C_1 always remains on C_1 and the branches are never interchanged. Hence $z = a$ is not a branch point. Obviously $W = \log z = \log |z| + i \arctan \frac{y}{x} + 2\pi n i$ has an infinitude of branches, and $z = 0$ is its branch point.

It was pointed out in § 112 that branch points are singular points of multiple-valued functions. The theorems of §§ 108–114 do not hold when branch points are enclosed by the contour of integration. This raises the question of the treatment and geometrical representation of multiple-valued functions of a complex variable. In the case of real functions, such as $y = \pm\sqrt{x}$, both branches of the function are drawn in the xy-plane for real values of x taken along the positive x-axis. In the case of multiple-valued complex functions the W-plane is made up of more than one plane or sheet; one sheet for each branch. These sheets are joined at the branch point. For instance, in the case of $W = \log z$ the branch point is $z = 0$ and the entire W-plane is represented by a helical surface such as a winding stairway which winds about the point $z = 0$. That is, since

$$W = \log \rho + i\theta,$$

as z describes a circle about $z = 0$ the value of ρ remains constant but $i\theta$ increases by $2\pi i$ with each revolution. If a plane, containing the axis of the spiral, extends outward on one side from this axis, it cuts the spiral surface into infinitely many sheets. If a contour is drawn on any one of these sheets in such a way that it does not cross the curve formed by the intersection of the plane and the spiral surface, then $\log z$, for the values of z on the contour, is a single-valued analytic function and the theorems so far proved apply on any single sheet.

The value of $p^n 1 (1 > n > 0)$ is frequently required. It has been shown in § 116 that (321) is the solution of the circuit equations for $\Phi(p)$ an algebraic function of p. It has been shown elsewhere (see Ref. 43 at end of text) that (321) is the solution of the circuit equation for $\Phi(p)$ an analytic function. Accordingly, by the Bromwich integral (321),

$$p^n 1 = \frac{1}{2\pi i}\int_C \lambda^{n-1} e^{\lambda t} d\lambda,$$

where C is the line described in exercise 13 § 122.

The integrand evidently has a branch point at $\lambda = 0$. However, on and within the contour shown in Fig. 73a the integrand is single-valued and analytic.

In most physical systems the real parts of the roots are negative, hence there are no singularities of the integrand in the right half of

the z-plane, and contour (a) is replaceable by (b) in Fig. 73. Applying Cauchy's theorem to the contour (b) we have

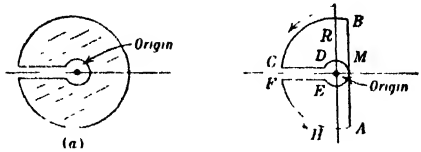


FIG. 73 Contour for Fractional Exponent

$$\underset{\text{(I)}}{\int_{AB}\lambda^{n-1}e^{\lambda t}d\lambda} + \underset{\text{(II)}}{\int_{(\pi/2)-\epsilon}^{\pi}(Re^{i\phi})^{n-1}e^{tR(\cos\phi+i\sin\phi)}R(-\sin\phi+i\cos\phi)d\phi}$$

$$+\underset{\text{(III)}}{\int_R^r(\rho e^{i\pi})^{n-1}e^{t\rho\cos\pi}e^{i\pi}d\rho}$$

$$+\underset{\text{(IV)}}{\int_{\pi}^{-\pi}(re^{i\phi})^{n-1}e^{tr(\cos\phi+i\sin\phi)}r(-\sin\phi+i\cos\phi)d\phi}$$

$$+\int_r^R(\rho e^{-i\pi})^{n-1}e^{t\rho\cos(-\pi)}e^{-i\pi}d\rho$$

$$+\underset{\text{(VI)}}{\int_{-\pi}^{-(\pi/2)+\epsilon}(Re^{i\phi})^{n-1}e^{tR(\cos\phi+i\sin\phi)}R(-\sin\phi+i\cos\phi)d\phi}$$

$$=0.$$

The limits on integrals III and V are positive quantities since the transformation on λ is $\lambda = \rho e^{i\theta}$. When $\lambda = -R + 0i$ and $\theta = \pi$, then $\rho = R$. As $R \to \infty$ and $r \to 0$, it is easily shown that integrals II, IV, and VI are zero. (See Ex. 11 at the end of this section.) Hence

$$\frac{1}{2\pi i}\int_{AB}\lambda^{n-1}e^{\lambda t}d\lambda = -\frac{1}{\pi}\int_0^{\infty}\rho^{n-1}\sin(n-1)\pi e^{-t\rho}d\rho$$

$$=\frac{\sin n\pi}{\pi}\int_0^{\infty}\rho^{n-1}e^{-t\rho}d\rho.$$

Let $\rho t = y$. Then

$$\frac{1}{2\pi i}\int_{AB}\lambda^{n-1}e^{\lambda t}d\lambda = \frac{\sin n\pi}{\pi t^n}\int_0^{\infty}y^{n-1}e^{-y}dy.$$

The last integral is a **gamma function** $\Gamma(n)$. It cannot be expressed in terms of a finite number of elementary (calculus) functions. But its value for all real values of n can be found in a table of gamma functions. Finally then

$$p^n 1 = \frac{\sin n\pi}{\pi t^n}\Gamma(n) \quad (1 > n > 0). \tag{329}$$

Evidently

$$\left.\begin{aligned} p^{n+1} 1 = p(p^n 1) &= \frac{d}{dt}\frac{(\sin n\pi)\Gamma(n)}{\pi t^n} \\ &= -\frac{n(\sin n\pi)\Gamma(n)}{\pi t^{n+1}}. \end{aligned}\right\} \tag{330}$$

In a similar way, $p^{n+r} 1$ (r a positive integer) can be written down at once.

120. Heaviside's Third Rule. Heaviside's third rule is: If $\Phi(p)$ can be written as

$$\Phi(p) = (a_0 + a_1 p + a_2 p^2 + \ldots) + p^{\frac{1}{2}}(b_0 + b_1 p + b_2 p^2 + \ldots)$$

then

$$\Phi(p)\, 1 = a_0 + \frac{1}{(\pi t)^{\frac{1}{2}}}\left(b_0 - \frac{b_1}{2t} + \frac{b_2 1\cdot 3}{(2t)^2} - \frac{b_3 1\cdot 3\cdot 5}{(2t)^3} + \ldots\right) \tag{331}$$

for $t > 0$. There has been much controversy over the validity of this rule. To see that it is true, it is necessary only to substitute $\Phi(p)$ in (321) and make use of (329), (330), and a table of gamma functions.

121. Heaviside's Extended Problem. Thus far we have been con-concerned only with Heaviside's unextended problem. The extended problem stated mathematically is: *Given the n simultaneous linear differential equations (319) with constant coefficients where the voltages $f_i(t)$ ($i = 1, 2, \ldots n$) are suddenly applied to the network at $t = 0$; one voltage in each mesh. At least one $f_i(t)$ is neither zero nor unit function. No charges exist and no currents are flowing at time $t = 0$. Find the response of the network.*

The procedure is as follows: First solve Eqs. (319) under the restriction that $f_i(t) = 0 (i = 2, 3, \ldots n)$ and $f_1(t) \neq 0$. Next let $f_i(t) = 0 (i = 1, 3, \ldots n)$ and $f_2(t) \neq 0$. Carry on this process, finally, solving (319) for all $f_i(t) = 0$ except $f_n(t)$. Now it is known from the theory of such differential equations that if the n values obtained for (say) i_k are added, the total value for i_k so obtained is identical to that obtained by solving (319) with no $f_i(t) = 0$. This is the well-known principle of **superposition.** Consequently it is necessary only to investigate the operational solution of

$$
\begin{aligned}
z_{11}(p)i_1 + \dots + z_{1n}(p)i_n &= f_1(t)\,\mathit{1}, \\
z_{21}(p)i_1 + \dots + z_{2n}(p)i_n &= 0, \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad&\cdot \\
z_{n1}(p)i_1 + \dots + z_{nn}(p)i_n &= 0,
\end{aligned}
\tag{332}
$$

subject to the initial conditions of equilibrium configuration. The symbol $f_1(t)\,\mathit{1}$ signifies that the voltage $f_1(t)$ is suddenly applied at $t = 0$, the previous voltage being zero.

Suppose that the indicial admittance $A(t)$ has been found; that is, Eqs. (318) have been solved subject to the initial conditions of the preceding paragraph. Then it may be shown that the current in the first mesh due to the suddenly applied voltage $\mathit{1}\,f(t)$ in that mesh is

$$
i(t) = f(0)A(t) - \int_0^t A(t-\lambda)f'(\lambda)d\lambda. \tag{333}
$$

This is Duhamel's superposition integral. In this formula $f'(\lambda)$ means $\dfrac{df(t)}{dt}$ with t then replaced by the real variable λ. Before proving (333) let us illustrate its use.

EXAMPLE 1. A circuit consisting of an inductance L and condenser C in series is connected at $t = 0$ to an alternator supplying voltage $E \cos \omega t$. Find the current. The indicial admittance is

$$
\begin{aligned}
A(t) &= \frac{1}{pL + \dfrac{1}{pC}}\,\mathit{1} = \frac{1}{L}\cdot\frac{p}{p^2 + a^2}\,\mathit{1} \\
&= \frac{1}{aL}\sin at,
\end{aligned}
$$

where

$$
a^2 = \frac{1}{LC}.
$$

Then

$$
f(t) = E\cos\omega t, \quad f'(\lambda) = -\omega E\sin\omega\lambda,
$$

and

$$
\begin{aligned}
i(t) &= \frac{E}{aL}\sin at + \int_0^t \frac{1}{aL}\sin a(t-\lambda)\cdot(-\omega E)\sin\omega\lambda d\lambda \\
&= \frac{E}{aL}\sin at + \frac{\omega E}{2aL}\int_0^t \cos(\omega\lambda - a\lambda + at)d\lambda
\end{aligned}
$$

$$-\frac{\omega E}{2aL}\int_0^t \cos(\omega\lambda + a\lambda - at)d\lambda$$

$$= \frac{E}{aL}\sin at + \frac{\omega E}{2aL}\left[\frac{\sin\omega t - \sin at}{\omega - a} - \frac{\sin\omega t + \sin at}{\omega + a}\right]$$

$$= \frac{E}{aL}\sin at + \frac{E\omega}{aL(\omega^2 - a^2)}(a\sin\omega t - \omega\sin at).$$

Relation (333) is derived by adding or superposing the currents due to each increment of voltage $f(t)$ from $t = 0$ to $t = t$. We are interested in the voltage at time t. If the voltage E is applied at time $t = 0$, the response is $EA(t)$. If another voltage E_1 is applied at time $t = \lambda$ the new portion of response is $E_1A(t - \lambda)$. The total current then at time t subsequent to $t = \lambda$ is $EA(t) + E_1A(t - \lambda)$. Consider now the voltage $f(t)$ represented by dotted curve in Fig. 74. Let this voltage be replaced temporarily by the stair-step voltage drawn in the solid line. The increment of voltage at $t = \lambda$ then is $\frac{df(t)}{dt}\Delta\lambda$. The current due to this increment is $A(t - \lambda)f'(\lambda)\Delta\lambda$. The current due to the sum of such increments is

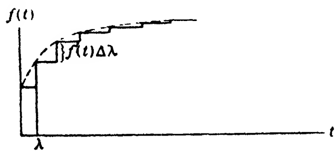


FIG. 74.

$$\sum A(t - \lambda_i)f'(\lambda_i)(\Delta\lambda)_i$$

As $\Delta\lambda \to 0$ and $n \to \infty$ the stair-step voltage approaches $f(t)$ and the sum becomes

$$\int_0^t A(t - \lambda)f'(\lambda)d\lambda.$$

There must be added to this current the current due to the voltage $f(0)$ applied at $t = 0$. Thus the total current is

$$i = f(0)A(t) + \int_0^t A(t - \lambda)f'(\lambda)d\lambda,$$

and (333) is established. Eq. (333) gives the total response, that is, transient plus steady-state current. If the current in mesh k due to impressed voltage in mesh j is desired then we have

$$i_k = f(0)A_{jk}(t) + \int_0^t A_{jk}(t - \lambda)f'(\lambda)d\lambda,$$

where A_{jk} is the transfer indicial admittance defined in § 115.

EXAMPLE 2. Consider the circuit shown in Fig. 75. Write the operational expression for the transfer indicial admittance.

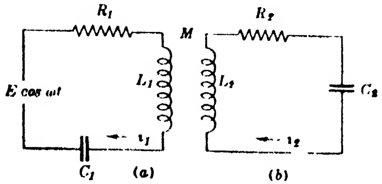


FIG. 75.—Transformer Circuit.

The differential equations are (if positive currents produce opposing fluxes)

$$\left(L_1p + R_1 + \frac{1}{C_1p}\right)i_1 - Mpi_2 = 1$$

$$-Mpi_1 + \left(L_2p + R_2 + \frac{1}{C_2p}\right)i_2 = 0,$$

$$i_2 = A_{12}(t) = \Phi(p)\,1 = \frac{\begin{vmatrix} L_1p + R_1 + \dfrac{1}{C_1p} & 1 \\ -Mp & 0 \end{vmatrix}}{\begin{vmatrix} L_1p + R_1 + \dfrac{1}{C_1p} & -Mp \\ -Mp & L_2p + R_2 + \dfrac{1}{C_2p} \end{vmatrix}}$$

If $\Phi(p)$ is substituted in (321), $A_{12}(t)$ is readily found as a function of the time t.

122. Summary. If only one voltage is impressed suddenly on a network the steps of the solution are as follows:

(*a*) Write the differential equations of the network by the principles of § 19 supposing the applied voltage to be $1E$.

(*b*) Solve for the current (say i_k) by determinants, obtaining

$$i_k = \frac{M_{1k}(p)}{D(p)}\,1E.$$

(*c*) Evaluate this operational expression by substituting in the Bromwich integral (321). If $1E$ was applied in the kth mesh, then we have the indicial admittance. If $1E$ is applied in some other mesh, the above expression is the transfer indicial admittance.

(*d*) Now use the superposition theorem (333) where the voltage if $f(t)$.

(*e*) If n voltages are applied, one in each mesh, repeat steps (*a*) to (*d*) for each voltage and add the resulting currents.

This short section thus contains the elements of the Heaviside operational calculus.

EXERCISES AND PROBLEMS

Establish, by means of the Bromwich line integral, the ten operational formulas following

1. $\dfrac{1}{p^m}\,1 = \dfrac{t^m}{m!}$ (m a positive integer).

2. $\dfrac{p}{(p+b)^2}\,1 = te^{-bt}$.

3. $\dfrac{p}{(p+a)(p+b)}\,1 = \dfrac{1}{a-b}(e^{-bt} - e^{-at})$.

4. $\dfrac{p^2}{(p+b)^2}\,1 = (1 - bt)e^{-bt}$.

5. $\dfrac{1}{p+a}\,1 = \dfrac{1}{a}(1 - e^{-at})$.

6. $\dfrac{1}{(p+a)^2}\,1 = \dfrac{1}{a^2}[1 - (1 + at)e^{-at}]$.

7. $\dfrac{p}{(p^2 + a^2)(p + a\tan\beta)}\,1 = \dfrac{\cos\beta}{a^2}[\cos\beta\; e^{-a\tan\beta t} - \cos(at + \beta)]$.

8. (*a*) $p^{1/2}\,1 = (\pi t)^{-1/2}$. (*c*) $p^{-1/2}\,1 = (4t/\pi)^{1/2}$.

(*b*) $p^{3/2}\,1 = -(4\pi t^3)^{-1/2}$. (*d*) $p^{-3/2}\,1 = \dfrac{4}{3}\left(\dfrac{t^3}{\pi}\right)^{1/2}$.

9. $\dfrac{p}{\sqrt{p^2 - a^2}}\,1 = \displaystyle\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(\frac{at}{2}\right)^{2n}$.

10. $\dfrac{\omega p}{(p+b)^2 + \omega^2}\,1 = e^{-bt}\sin\omega t$.

11. Show that integrals II, IV, and VI of § 119 are zero.

12. A voltage E is suddenly impressed upon the primary circuit of a transformer. Find the currents i_1 and i_2 at any time thereafter when $E = 100$ volts, $L_1 = 1.0$ henry, $L_2 = 2.0$ henrys, $M = 0.5$ henry, $R_1 = 10.0$ ohms, and $R_2 = 5.0$ ohms.

13. Establish the Bromwich integral Eq. (321), § 116, where the contour is a line from $-i\infty$ to $+i\infty$ such that all singularities of the integrand lie to the left of this line.

14. Repeat the reasoning of § 116 if every z_{rs} in Eqs. (318) is replaced by z_{rs}^m, where m is a positive integer

15. Repeat the reasoning of § 116 if the unit function is defined to be a voltage which is zero for $t < 0$, equal to every value from 0 to 1 inclusive at $t = 0$ and equal to unity for $t > 0$.

III

MANIPULATIVE DEVICES AND APPLICATIONS OF THE OPERATIONAL CALCULUS

The last section contains all the essentials of the operational calculus. The fact that (321) is valid for $\Phi(p)$ any analytic function is a result of great importance. However, there exist certain tricks or devices which frequently lessen the labor of obtaining results. It is the purpose of this division to explain some of the most important of these and to illustrate applications of the operational calculus in the analysis of transmission lines, the study of heat flow in refrigerator walls, and the design of brake shoes for the rotors of large generators.

123. Algebraic Operations and Shifting. Although the Bromwich integral always gives the correct answer to the problem, it is sometimes convenient to evaluate operational expressions by other methods. For example, the expression can often be simplified materially before being substituted in this integral; in some cases, it may even be simplified to such an extent that it can be recognized as a standard form. A number of standard forms or formulas are found in § 122. Two simplifying processes of particular importance are the algebraic transformations of the operator and the so-called "shifting" processes. These are now considered.

In the development of the Bromwich integral, it was shown that it is legitimate to perform any algebraic operations on $\Phi(p)$ that are valid for the integrand of (321). Two such operations that are frequently useful are expansion into series and decomposition into partial fractions. Examples of series expansions were given in Heaviside's second and third rules above.

EXAMPLE 1. Evaluate $\dfrac{p}{p^2 + a^2}1$ by the method of partial fractions. Since in the integrand of the Bromwich integral (321)

$$\frac{\lambda}{\lambda^2 + a^2} = \frac{1}{2}\left(\frac{1}{\lambda + ia} + \frac{1}{\lambda - ia}\right)$$

we may write

$$\frac{}{p^2 + a^2} = \frac{}{(p + ia)(p - ia)} = \frac{1}{2}\left(\frac{1}{p + ia} + \frac{1}{p - ia}\right).$$

The last two fractions are immediately evaluated by means of the formulas of § 122. Whence

$$\frac{p}{p^2 + a^2} 1 = \frac{1}{2}\left[\frac{1}{ia}(1 - e^{-iat}) - \frac{1}{ia}(1 - e^{+iat})\right]$$

$$= \frac{e^{iat} - e^{-iat}}{2ia} = \frac{1}{a}\sin at.$$

By means of the "shifting" process, a factor may be removed from an operational expression before evaluation. If the presence of a certain factor in the final result is suspected, it is natural to suppose that the removal of this factor will simplify the evaluation of the operational expression which remains.

In the Bromwich integral (321), multiply by e^{at} and obtain

$$e^{at}\Phi(p)1 = \frac{1}{2\pi i}\int_c \frac{\Phi(\lambda)e^{(\lambda+a)t}d\lambda}{\lambda}.$$

In order to integrate this expression, change the variable from λ to λ', where $\lambda = \lambda' - a$. Then

$$e^{at}\Phi(p)1 = \frac{1}{2\pi i}\int_c \frac{\Phi(\lambda' - a)e^{\lambda' t}d\lambda'}{\lambda' - a}$$

$$= \frac{1}{2\pi i}\int_c \frac{\frac{\lambda'}{\lambda' - a}\Phi(\lambda' - a)e^{\lambda' t}d\lambda'}{\lambda'}$$

$$= \frac{p}{p - a}\Phi(p - a)\, 1.$$

Therefore,

$$\Phi(p)1 = e^{-at}\frac{p}{p - a}\Phi(p - a)\, 1.$$

This formula shows the form taken by $\Phi(p)$ when an exponential e^{-at} is factored out of the final result.

EXAMPLE 2. Evaluate $\frac{p}{(p + a)^n} 1$. Here, owing to the presence of the quantity $p + a$, we suspect e^{-at} as a factor. The removal of this factor gives

$$\Phi(p)1 = \frac{p}{(p + a)^n} 1 = e^{-at}\left(\frac{p}{p - a}\cdot\frac{p - a}{p^n} 1\right) = e^{-at}(p^{1-n})1.$$

Finally, by Ex. 1, § 122, we have

$$\Phi(p)\,1 = e^{-at}\frac{t^{n-1}}{(n-1)!}\ \text{(for } n-1 \text{ a positive integer).}$$

Example 3. Evaluate again $\dfrac{p}{p^2+a^2}\,1$. The denominator may be written $(p+ia)(p-ia)$, where we may suspect the presence of a factor e^{-iat}. Its removal gives

$$\frac{p}{(p+ia)(p-ia)}\,1 = e^{-iat}\frac{p}{p-ia}\cdot\frac{p-ia}{p(p-2ia)}\,1$$

$$= e^{-iat}\frac{p}{p-2ia}\,1$$

$$= e^{-iat}\frac{1-e^{2iat}}{-2ia} = \frac{1}{a}\sin at,$$

from Ex. 5, § 122. In this example, the final result did not contain the suspected factor e^{-iat}, but its removal simplified the solution.

By means of the Bromwich integral we may examine the effects of factors of $\Phi(p)$ such as $p+a$ and $\dfrac{1}{p+a}$.

$$(p+a)\Phi(p)\,1 = \frac{1}{2\pi i}\int_C \frac{(\lambda+a)\Phi(\lambda)e^{\lambda t}d\lambda}{\lambda}$$

$$= \frac{e^{-at}}{2\pi i}\int_C \frac{(\lambda+a)\Phi(\lambda)e^{(\lambda+a)t}d\lambda}{\lambda}$$

$$= \frac{e^{-at}}{2\pi i}\frac{d}{dt}\int_C \frac{\Phi(\lambda)e^{(\lambda+a)t}d\lambda}{\lambda}$$

$$= e^{-at}\frac{d}{dt}\,[e^{at}\Phi(p)\,1].$$

In particular, if $a = 0$,

$$p\Phi(p)\,1 = \frac{d}{dt}\,[\Phi(p)\,1],$$

thus identifying p as a differentiating operator. If a is not zero, then by performing the indicated differentiation,

$$(p+a)\Phi(p)\,1 = \frac{d}{dt}\,[\Phi(p)\,1] + a\Phi(p)\,1,$$

thus illustrating again the fact stated previously, that algebraic operations on $\Phi(p)$ are permissible.

Again,

$$\frac{1}{p+a}\Phi(p)1 = \frac{1}{2\pi i}\int_C \frac{\Phi(\lambda)e^{\lambda t}d\lambda}{(\lambda+a)\lambda}$$

$$= \int_{-\infty}^{t}\left[\frac{1}{2\pi i}\int_C \frac{\Phi(\lambda)e^{\lambda t}d\lambda}{\lambda+a}\right]dt$$

since

$$\frac{e^{\lambda t}}{\lambda} = \int_{-\infty}^{t} e^{\lambda t}dt.$$

Change the variable from λ to λ', where $\lambda = \lambda' - a$. Then

$$\frac{1}{p+a}\Phi(p)1 = \int_{-\infty}^{t}\left[\frac{1}{2\pi i}\int_C \frac{\Phi(\lambda'-a)e^{\lambda' t}e^{-at}d\lambda'}{\lambda'}\right]dt$$

$$= \int_{-\infty}^{t} e^{-at}\,[\Phi(p-a)1]\,dt.$$

Since $\Phi(p-a)1$ is zero when $t < 0$, the integration from $-\infty$ to 0 therefore yields zero, and the result may be written

$$\frac{1}{p+a}\Phi(p)1 = \int_{0}^{t} e^{-at}\,[\Phi(p-a)1]dt.$$

In particular, if $a = 0$,

$$\frac{1}{p}\Phi(p)1 = \int_0^t \Phi(p)1dt,$$

thus identifying $1/p$ as an integrating operator.

EXAMPLE 4. Evaluate $(p-a)^{-3}1$. By means of the formula just established,

$$\frac{1}{(p+a)^3}1 = \int_0^t e^{-at}\frac{1}{p^2}1dt = \frac{1}{2}\int_0^t t^2e^{-at}dt$$

$$= \frac{2}{a_3}[1 - (1 + at + \tfrac{1}{2}a^2t^2)e^{-at}].$$

124. The Derivation of the Partial Differential Equations of the Transmission Line. The differential equations, § 19, for circuits with

concentrated parameters (inductance, capacitance, and resistance) are ordinary differential equations. On the other hand, the equations for the transmission line current and voltage, since the parameters are distributed, are partial differential equations. These equations may be derived by applying Kirchhoff's laws (§ 19) to an element of length of the line. A single line with ground return is considered.

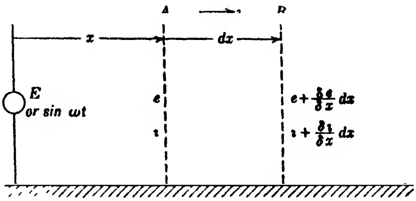


FIG. 76.—Transmission Line.

Accordingly, referring to Fig. 76, the voltage above ground at A plus the rise of voltage in the infinitesimal length dx is equal to the voltage above ground at B.

Let the notation be:

e = voltage at a distance x from the sending end of the line.
i = current above ground at a distance x from the sending end of the line.

C = capacitance
R = resistance
L = inductance
G = leakage conductance

per unit length of one wire of the line.

Then, if the voltage at A is e, at B, $e + \frac{\partial e}{\partial x} dx$, and the rise of voltage is $-\left(Rdxi + Ldx\frac{di}{dt}\right)$,

$$e - Ridx - L\frac{di}{dt}dx = e + \frac{\partial e}{\partial x}dx,$$

or

$$\frac{\partial e}{\partial x}dx = - Ridx - L\frac{di}{dt}dx.$$

In a similar way, by Kirchhoff's second law, the current equation is

$$-\frac{\partial i}{\partial x}dx = Gedx + C\frac{de}{dt}dx.$$

These equations evidently reduce to

$$
\begin{aligned}
\frac{\partial e}{\partial x} &= -(R + Lp)i, \\
\frac{\partial i}{\partial x} &= -(G + Cp)e,
\end{aligned}
\tag{334}
$$

where $p = \frac{d}{dt}$. These are the transmission line equations.

By differentiation and substitution, Eqs. (334) become

$$
\begin{aligned}
\frac{\partial^2 i}{\partial x^2} - n^2 i &= 0, \\
\frac{\partial^2 e}{\partial x^2} - n^2 e &= 0,
\end{aligned}
\tag{335}
$$

where $n^2 = (R + Lp)(G + Cp)$. By § 11, the solutions of (335) are seen to be

$$
\begin{aligned}
i &= e^{-nx}K_1 + e^{nx}K_3, \\
e &= e^{-nx}K_2 + e^{nx}K_4,
\end{aligned}
\tag{336}
$$

where the K's, which are either constants or functions of the time only, are determined by the conditions of the particular transmission line problem to be solved.

125. Transmission Line Problems. Owing to various terminal conditions on transmission lines and the different assumptions made regarding the relative magnitude of the distributed constants, Eqs. (336) lead to a very large number of different transmission line problems. From the operational point of view, these problems have certain features in common. In general, the operator p is involved, both irrationally and transcendentally, in the operational expressions resulting from (336). The total current, both transient and steady-state, can be obtained from the indicial admittance by use of Duhamel's superposition theorem. Two particular problems are now considered illustrating these features.

EXAMPLE 1. *Infinite distortionless line.* Let it be required to find the transmission line current at any point x and at any time t due to impressed unit or sinusoidal voltage, when the constants of the line are such that $RC = GL$. Such a transmission line is said to be distortionless. If the line is of infinite length, the current and voltage

will both decay for x sufficiently large and hence $K_3 = K_4 = 0$ in Eqs. (336). Thus

$$i = e^{-nx}K_1$$

and

$$e = e^{-nx}K_2,$$

so that, using (334),

$$\frac{\partial i}{\partial x} = -\, ne^{-nx}K_1 = -\,(G + Cp)e = -\,(G + Cp)e^{-nx}K_2.$$

Therefore,

$$K_2 = \frac{n}{G + Cp}K_1 = \sqrt{\frac{R + Lp}{G + Cp}}\,K_1.$$

When $x = 0$, $i = K_1$, and it is seen that K_1 is the current entering the line. Likewise, K_2 is the impressed voltage. If the unit e.m.f. is impressed, the entering current i_0 will be given by

$$i_0 = \sqrt{\frac{G + Cp}{R + Lp}}\,1$$

while the current at any point along the line is

$$i = e^{-nx}\sqrt{\frac{G + Cp}{R + Lp}}\,1$$

and the voltage at any point is

$$e = e^{-nx}\,1.$$

Since $RC = GL$

$$n = \sqrt{\frac{C}{L}}\,(R + Lp)$$

and

$$i_0 = \sqrt{\frac{C}{L}}\,1 = \sqrt{\frac{C}{L}},$$

so that the entering current is steady. The current at a distance x down the line is

$$i = \sqrt{\frac{C}{L}}\,e^{-\sqrt{C/L}\,(R+Lp)x}\,1$$

$$= \sqrt{\frac{C}{L}}\,e^{-Rx\sqrt{C/L}}\,e^{-x\sqrt{LC}\,p}\,1.$$

The operator $e^{-x\sqrt{LC}p}\,1$ which occurs here may be evaluated by means of the Bromwich integral:

$$e^{-ap}1 = \frac{1}{2\pi i}\int_C \frac{e^{\lambda(t-a)}d\lambda}{\lambda}.$$

This is the form of the unit function with $t - a$ instead of t. Its value is therefore zero when $t - a < 0$ and unity when $t - a > 0$. The solution of our problem is then

$$i = 0 \qquad \text{for} \quad t < x\sqrt{LC},$$

$$i = \sqrt{\frac{C}{L}}\,e^{-\sqrt{GRx^2}} \qquad \text{for} \quad t > x\sqrt{LC}.$$

The current at any point is thus zero until the current "wave" has had time to reach the point, after which it remains at a constant value. The propagation velocity is

$$v \equiv \frac{x}{t} = (LC)^{-\frac{1}{2}}.$$

By means of the superposition theorem, we may investigate the effect of impressing a voltage $e_0 = E \sin \omega t$ at the terminals. Here

$$e(\lambda) = E \sin \omega\lambda, \qquad e(0) = 0,$$

$$A(t) = 0 \qquad \text{for} \qquad t < x\sqrt{LC},$$

$$A(t-\lambda) = 0 \qquad \text{for} \qquad \lambda > t - x\sqrt{LC},$$

$$A(t) = \sqrt{\frac{C}{L}}\,e^{-\sqrt{GRx^2}} \qquad \text{for} \qquad t > x\sqrt{LC},$$

$$A(t-\lambda) = \sqrt{\frac{C}{L}}\,e^{-\sqrt{GRx^2}} \qquad \text{for} \qquad \lambda < t - x\sqrt{LC}.$$

Then by (333)

$$i(t) = e(0)A(t) + \int_0^t A(t-\lambda)e'(\lambda)d\lambda$$

$$= \int_0^{t-x\sqrt{LC}} A(t-\lambda)e'(\lambda)d\lambda + \int_{t-x\sqrt{LC}}^{t} A(t-\lambda)e'(\lambda)d\lambda.$$

In the first of these integrals, $A(t - \lambda)$ is a constant, while in the second it is zero. Hence

$$i(t) = \sqrt{\frac{C}{L}}\,e^{-\sqrt{GRx^2}}e(\lambda)\Big|_0^{t-x\sqrt{LC}}$$

This formula shows the form of the current due to any impressed voltage $e(t)$. In particular, for the sinusoid,

$$i(t) = E\sqrt{\frac{C}{L}}\, e^{-\sqrt{GRx^2}} \sin \omega(t - x\sqrt{LC}).$$

Thus, irrespective of the frequency the current wave is attenuated the same amount for a given distance x. It is for this reason that the line is called distortionless.

EXAMPLE 2. *Open-circuit ideal cable.* Let it be required to find the voltage and current at time t at any point x of the open circuited ideal cable of Fig. 77. Essentially, an ideal cable is a transmission line which has insignificant leakage and inductance.

In Fig. 77 the distance x is measured from the receiving end of the line, instead of from the sending end as in Fig. 76. Consequently it is easily verified that all algebraic signs in the differential equations corresponding to Eqs. (334) are positive. Eqs. (335–336), however, are unchanged.

FIG. 77.

From the general definitions of e^z, sin z, and cos z (§ 109) and from the definitions of cosh x and sinh x given in the calculus, it readily follows that

$$e^{nx} = \cosh nx + \sinh nx,$$

$$e^{-nx} = \cosh nx - \sinh nx.$$

Consequently the second of Eqs. (336) may be written

$$\left.\begin{aligned} e &= (K_4 + K_2)\cosh nx + (K_4 - K_2)\sinh nx \\ &= A_1 \cosh nx + A_2 \sinh nx. \end{aligned}\right\} \quad (337)$$

Then by Eqs. (334)

$$\frac{\partial i}{\partial x} = (G + Cp)e = (G + Cp)(A_1 \cosh nx + A_2 \sinh nx),$$

or

$$i = \frac{(G + Cp)}{}(A_1 \sinh nx + A_2 \cosh nx). \quad (338)$$

Since $G = L = 0$ (ideal cable),

$$n = \sqrt{RCp}$$

and

$$i = \sqrt{\frac{Cp}{R}}\,(A_1 \sinh x\sqrt{RCp} + A_2 \cosh x\sqrt{RCp}).$$

Let the applied voltage be $\mathit{1}E$. Since the line is open-circquited, it follows that

$$i = 0 \qquad \text{for} \qquad x = 0,$$

$$e = \mathit{1}E \qquad \text{for} \qquad x = l.$$

Substituting these boundary conditions in Eqs. (337–338), it is obvious that

$$A_2 = 0, \; e = A_1 \cosh nx, \text{ and } E\mathit{1} = A_1 \cosh nl.$$

Finally,

$$e = \frac{E \cosh nx}{\cosh nl}\,\mathit{1}, \tag{339}$$

$$i = E\left(\frac{Cp}{R}\right)^{\frac{1}{2}} \frac{\sinh nx}{\cosh nl}\,\mathit{1}. \tag{340}$$

Evaluating (340) by (321), we have

$$i = E\left(\frac{Cp}{R}\right)^{\frac{1}{2}} \frac{\sinh nx}{\cosh nl}\,\mathit{1} = \frac{E}{2\pi j}\left(\frac{C}{R}\right)^{\frac{1}{2}} \int_C \frac{\lambda^{\frac{1}{2}} \sinh (x\sqrt{RC\lambda})}{\lambda \cosh (l\sqrt{RC\lambda})} e^{\lambda t} d\lambda,$$

(where now $j = \sqrt{-1}$). Let the variable of integration be changed from λ to z by $\lambda = z^2$. Then

$$i = \frac{E}{\pi j}\left(\frac{C}{R}\right)^{\frac{1}{2}} \int_{C_1} \frac{\sinh (xz\sqrt{RC})}{\cosh (lz\sqrt{RC})} e^{z^2 t} dz.$$

The variable z runs over the semicircle C_1 (Fig. 78) in the upper half-plane as λ traverses the infinite circle C. But C_1 is not a closed contour and consequently the residue theorem does not apply. Since there are no branch points of the integrand within the contour let $C = C_2$ be a circle twice drawn. The value for i then becomes

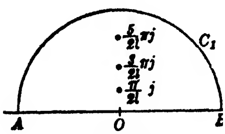


FIG. 78.

$$i = \frac{E}{4\pi j}\left(\frac{C}{R}\right)^{\frac{1}{2}} \int_{C_2} \frac{\lambda^{\frac{1}{2}} \sinh (x\sqrt{RC\lambda})}{\lambda \cosh (l\sqrt{RC\lambda})} e^{\lambda t} d\lambda.$$

Let $\lambda \quad \overline{RC}\cdot$ The value for i then is

$$i = \frac{Ej}{2\pi R}\int_{C_3} \frac{\sin xu}{\cos lu}\cdot e^{-(u^2/RC)t}du,$$

where C_3 is a circle enclosing the roots of $\cos lu$. These roots are

$$u = \frac{(2s-1)}{2}\frac{\pi}{l} \qquad (s = 0, \pm 1, \pm 2, \ldots).$$

The residues of the integrand, by Ex. 1 of Sec. I of this chapter, are

$$R_s = \quad \operatorname{Lim} \frac{\left[u - \frac{(2s-1)}{2}\frac{\pi}{l}\right]\sin xu \cdot e^{-(u^2/RC)t}}{\cos lu} \qquad (s = 0, \pm 1, \pm 2, \ldots).$$

$$\frac{\sin \frac{(2s-1)}{2}\frac{\pi}{l}x \quad \left(\frac{2s-1}{2}\frac{\pi}{l}\right)^2 t}{l \sin \frac{(2s-1)\pi}{}} \qquad (s = 0, \pm 1, \pm 2, \ldots)$$

Finally by the residue theorem, § 114,

$$i = \frac{E}{Rl}\sum_{s=-\infty}^{s=+\infty}(-1)^{s+1}\left[\sin\frac{2s-1}{2}\frac{\pi}{l}x\right]\cdot \quad \left(\frac{2s-1}{2}\frac{\pi}{l}\right)^2 t$$

$$= \frac{2E}{Rl}\sum_{s=1}^{s=\infty}(-1)^{s+1}\left[\sin\frac{2s-1}{2}\frac{\pi}{l}x\right]\cdot \epsilon \quad \left(\frac{2s-1}{2}\frac{\pi}{l}\right)^2 t$$

In a similar manner, it can be shown that

$$e = E \quad 1 + \frac{4}{}\sum \frac{(-1)^s \cos \frac{2s-1}{2}\frac{\pi x}{l}}{2s-1} \quad \frac{\left(\frac{2s-1}{2}\frac{\pi}{l}\right)^2 t}{}$$

126. Partial Differential Equations of Linear Heat Flow. It may be seen that the differential equations for the flow of heat in one direction are precisely Eqs. (335), provided $L = C = 0$, and e and i denote respectively temperature and flux of heat at the point A in Fig. 79.

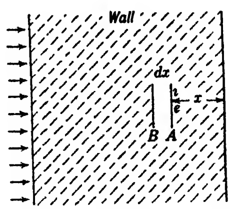


FIG. 79.—Refrigerator Wall.

Let the parameters pertaining to the flow be:

R = thermal resistance per unit length,

C = heat capacity, that is, the heat absorbed by the material per unit length per unit increase in temperature.

Let the temperature drop be taken positive in the direction of heat flow. Consider an infinitesimal length dx of the material. If the temperature at A is e, then the temperature at B is $e + \frac{\partial e}{\partial x} dx$. The temperature drop across dx is $Ridx$. Hence,

$$\frac{\partial e}{\partial x} dx = Ridx.$$

By similar reasoning

$$\frac{\partial i}{\partial x} dx = C \frac{\partial e}{\partial t} dx.$$

The equations of linear heat flow are

$$\left.\begin{aligned} \frac{\partial e}{\partial x} &= Ri, \\ \frac{\partial i}{\partial x} &= C \frac{\partial e}{\partial t}, \end{aligned}\right| \tag{341}$$

or

$$\frac{\partial^2 e}{\partial x^2} - RCpe = 0,$$

$$\frac{\partial^2 i}{\partial x^2} - RCpi = 0.$$

127. Refrigerator Box Heat Leakage. The transmission line theory so far developed is of great value in calculating refrigerator box heat leakage. Consider a particular case. To correlate calculations and test, an experimental box was built having the following specifications. The inside dimensions were 5 by 5 by 4 ft. with cork walls having a thickness of 4 in. Both the inside and outside of the box were lined with $\frac{1}{16}$-in. sheet steel. The fits between the inside lining and the cork filler and the outside lining were tight, so that no air spaces existed at any point between the two layers of the metal. The cork used in the box had a conductivity of 6 B.t.u. per sq. ft. of surface per in. of thickness per degree F. per 24 hr. Its density was 10 lb. per cu. ft., and its specific heat was 0.485.

One of the check tests was conducted as follows. An incandescent lamp rated at 100 watts was suspended at the center of the volume of the box. At the initial instant, the box was at the ambient temperature 70° F. By use of thermocouples, data were obtained for a curve between the temperature of the inner lining of the box and the time,

measuring time from the instant that voltage was applied to the lamp. The outside of the box was 70° F.

Let it be required to calculate a temperature-time curve for the inner lining of this box and plot the results up to the point at which the temperature reaches an approximately constant value.

For a first approximate result, it is sufficient to consider the heat density to be applied uniformly over the inner wall of the box. The effect of corners is neglected. The metal linings are also neglected. Fig. 79 and Eqs. (341) are respectively the figure and differential equations of the problem. By the method of §125, the general solution of the equations is

$$\theta = A_1 \cosh nx + A_2 \sinh nx$$
$$i = \frac{pC}{n}(A_1 \sinh nx + A_2 \cosh nx), \tag{342}$$

where $n = (RCp)^{1/2}$ and θ is temperature difference of the point whose coordinate is x and the exterior wall. Evidently, from the conditions of the problem, $\theta = 0$ for $x = 0$, and $i = i_0$ for $x = l$; θ_0 being the temperature of the exterior lining of the box, and i_0 the flux of heat applied to the interior lining.

Substituting these terminal conditions in (342), there results

$$\theta = \frac{ni_0}{pC}\frac{\sinh nx}{\cosh nl}.$$

This expression is evaluated by the Bromwich integral (321) as follows:

$$\frac{1}{4\pi j}\int_c \frac{ni_0}{\lambda^2 C}\frac{\sinh nx}{\cosh nl}e^{\lambda t}d\lambda,$$

where c is a circular contour twice drawn.

As in § 125 let $\lambda^{1/2} = \frac{ju}{(RC)^{1/2}}$ or $\lambda = \frac{-u^2}{RC}$. Then

$$\theta = \frac{1}{2\pi j}\int_{c_1} \frac{Ri_0 \sin ux}{u^2 \cos ul} e^{-u^2 t/RC} du,$$

where c_1 is a circular contour. The roots of the denominator within the contour evidently are $u = 0$ and

$$u = \frac{2s+1}{2}\frac{\pi}{l}, \quad s = 0, \pm 1, \pm 2, \ldots.$$

The residue at the double pole $u = 0$ is Ri_0x. The residues at the remaining poles within the contour are

$$R_s = Ri_0 \frac{(-1)^{s+1} \sin \frac{2s+1}{2}\frac{\pi}{l}x}{\left(\frac{2s+1}{2}\right)^2 \frac{\pi^2}{l}} e^{-\frac{1}{RC}\left(\frac{2s+1}{2}\right)^2 \frac{\pi^2}{l^2}t}, \quad s = 0, \pm 1, \pm 2, \ldots .$$

Finally, by the residue theorem of § 114,

$$\theta = Ri_0x - \frac{Ri_0l}{\pi^2} \sum \frac{(-1)^s}{\left(\frac{2s+1}{2}\right)^2} \sin \frac{2s+1}{2}\frac{\pi}{l}x \quad e^{-\frac{1}{RC}\left(\frac{2s+1}{2}\right)^2 \frac{\pi^2 t}{l^2}}.$$

To obtain the temperature at the interior surface, set $x = l$. We then have

$$\theta(l) = Ri_0l - \frac{2Ri_0l}{\pi^2}\sum_{s=0}^{\infty}\left(\frac{2}{2s+1}\right)^2 e^{-\frac{1}{RC}\left(\frac{2s+1}{2}\right)^2 \frac{\pi^2}{l^2}t}.$$

The constants of the solution are:

$$C = 0.485 \frac{\text{B.t.u.}}{\text{lb. °F.}} = 4.85 \frac{\text{B.t.u.}}{\text{ft.}^3 \text{ °F.}}$$

$$K = \frac{1}{R} = \frac{6 \text{ B.t.u. in.}}{\text{ft.}^2 \text{ day °F.}} = \frac{6 \text{ B.t.u. } \frac{1}{12} \text{ ft.}}{\text{ft.}^2 \text{ 86,400 sec. °F.}} = \frac{1}{172{,}800} \frac{\text{B.t.u.}}{\text{ft. sec. °F.}}$$

$$RC = 172{,}800 \frac{\text{ft. sec. °F.}}{\text{B.t.u.}} \times 4.85 \frac{\text{B.t.u.}}{\text{ft.}^2 \text{ °F.}} = 838{,}000 \frac{\text{sec.}}{\text{ft.}^2}$$

$$RCl^2 = 838{,}000 \frac{\text{sec.}}{\text{ft.}^2} \times \left(\frac{4}{12}\right)^2 \text{ft.}^2 = 9430 \text{ sec.}$$

$$i_0 \text{ (area)} = 100 \text{ watts} \times 0.000949 \frac{\text{B.t.u.}}{\text{watt sec.}} = 0.0949 \frac{\text{B.t.u.}}{\text{sec.}}$$

$$i_0 = \frac{0.0947 \text{ B.t.u. per sec.}}{140 \text{ ft.}^2} = 0.000678 \frac{\text{B.t.u.}}{\text{ft.}^2 \text{ sec.}}$$

$$Ri_0l = 172{,}800 \frac{\text{ft. sec. °F.}}{\text{B.t.u. ft.}} \times 0.000678 \frac{\text{B.t.u.}}{\text{ft.}^2 \text{ sec.}} \times \frac{1}{3} \text{ ft.} = 39.1\text{°F.}$$

Then the temperature difference $\theta(l)$ of the exterior and interior walls is

$$e(l) = 39.1\text{°F.} - 7.92\text{°F.} \sum_{s=0}^{\infty}\left(\frac{2}{2s+1}\right)^2 e^{-\left(\frac{2s+1}{2}\right)^2 \frac{t}{9430}}.$$

The curve of θ as a function of the time is shown in Fig. 80.

128. Water-wheel Generator Brake. Water-wheel generators are frequently equipped with friction brakes to stop the rotor and hold it against the torque due to leakage through the water gates. These brakes consist of a number of brake shoes disposed so as to apply pressure at a number of points around the horizontal surface of the rotor, rubbing on an annular steel plate, called the brake plate, bolted to the spider. The heat generated by the friction of the brake raises the temperature of the brake plate, setting up severe internal stresses which on several occasions have resulted in fractures. In designing such a brake, some questions which arise are: how thick should the

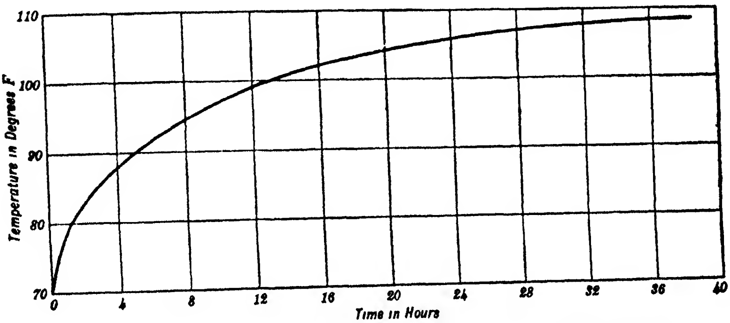


FIG. 80.—Temperature Difference of Interior and Exterior Wall.

brake plate be made, how much pressure should be applied on the brake shoes, and what are the maximum temperatures in the brake plate under various conditions?

The machine to be considered here is a 68-pole, 88.3-r.p.m., vertical water-wheel generator. The rotor consists of a spider built up of steel plates bolted together, with the field poles dovetailed around the periphery. The brake plate is bolted to the under side of the spider. It has a mean radius of 171 in. and a radial width of rubbing surface of 10 in. There are 20 brake shoes each 9 by 18 in. spaced equally around the plate and slightly staggered radially so as to completely utilize the 10-in. rubbing surface. Each shoe may be pushed upward against the brake plate by means of two hydraulic cylinders 6 in. in diameter. The coefficient of friction on the braking surface is 0.25.

The brake shoes are made of an interwoven copper-mesh asbestos compound. The brake plate is made of steel with the following thermal properties:

$$\text{Conductivity} = 0.46 \text{ watt per cm.}^2 \text{ per °C. per cm.}$$
$$\text{Specific heat} = 3.9 \text{ joules per cm.}^3 \text{ per °C.}$$

The WR^2 of the rotor is 1.69×10^8 lb-ft.^{2} The brake is applied at full speed, and the pressure in the operating cylinders remains constant. There is a gate-leakage torque of 200,000 lb-ft. which is approximately 3 per cent of full-load torque. The gate-leakage torque may be assumed not to vary with speed.

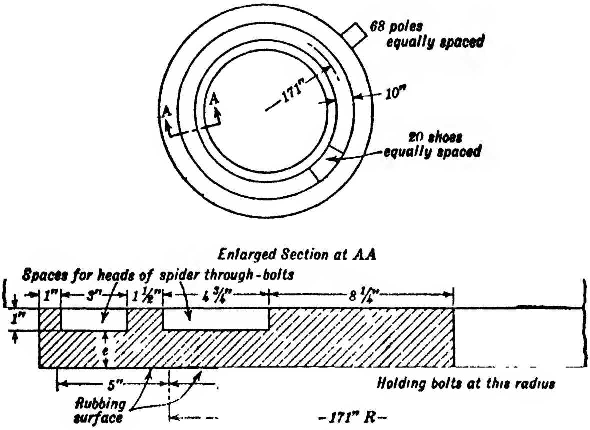


FIG. 81.—Brake Plate and Shoes.

EXAMPLE. Let it be required to determine the relation between maximum brake surface temperature and brake cylinder pressure.

Solutions, of course, may be made based on various assumptions. Since the brake plate is about 2 in. thick and the heat is applied for about 2 minutes at most, we first consider the problem equivalent to an infinite transmission line problem. It is assumed that the loss of heat from the sides of the brake plate is negligible.

The flux of heat or heat density applied is variable. The surface temperature, due to the application of this variable heat density, is calculated by means of the superposition theorem, Eq. (333), after the surface temperature due to a suddenly impressed constant heat density has been determined.

Referring to Fig. 76, and the first equations of example 1, § 125, we have

$$i = e^{-nx}K_1$$

and

$$e = e^{-nx}\,K_2,$$

where, from the above assumptions, $G = 0$ and $n = \sqrt{RCp}$. Let i_0 1 be the heat applied per unit area at time $t = 0$ and e_0 be the surface

temperature at $x = 0$. From Eqs. (334), $\frac{\partial i}{\partial x} = - Cpe$. Substituting the value of e above in this differential equation and integrating with respect to x, we have

$$i = K_2 \sqrt{\frac{Cp}{R}}\, e^{-n}$$

By the initial conditions of the problem, that is, $i = i_0\, 1$ for $x = t = 0$, there results

$$i_0\, 1 = K_2 \sqrt{\frac{Cp}{R}} \quad \text{or} \quad K_2 = \sqrt{\frac{R}{Cp}}\, i_0\, 1,$$

and hence the temperature at the braking surface is

$$e_0 = \sqrt{\frac{R}{Cp}}\, i_0\, 1.$$

By the Bromwich integral Eq. (321),

$$e_0 = 2\sqrt{\frac{Rt}{\pi C}}\, i_0.$$

The "indicial admittance" then is

$$e_0 = 2\sqrt{\frac{Rt}{\pi C}}\, 1.$$

The law of heat generation must be such that $i = i_0$ for $t = 0$ and $i = 0$ for $t = t_0$, where t_0 is the time required to stop the machine. Moreover, the power transfer into the heat band is assumed to be linear. Consequently,

$$i = \frac{i_0(t_0 - t)}{t_0}$$

By use of (333) we have

$$e_0 = 2\sqrt{\frac{R}{C\pi}}\, i_0 t^{1/2} + \int_0^t 2\sqrt{\frac{R}{\pi C}}\,(t - \lambda)^{1/2}\left(-\frac{i_0}{t_0}\right) d\lambda,$$

or

$$e_0 = \sqrt{\frac{R}{C\pi}}\,\frac{i_0}{t_0}\, t^{1/2}\left(2t_0 - \frac{4}{3}t\right).$$

The surface temperature e_0 is a maximum for any particular pressure when $t = \frac{t_0}{2}$; hence,

$$\text{Max. } e_0 = \frac{2}{3}\, i_0 \sqrt{\frac{2Rt_0}{\pi C}}. \tag{343}$$

It is now necessary to derive an expression for the total energy to be dissipated. This energy evidently is the kinetic energy of the rotor plus the energy generated during braking due to leakage.

$$\text{K.E. of the rotor} = \frac{1.69 \times 10^8(9.24)^2}{32.2} = 4.48 \times 10^8 \text{ ft-lb.}$$

$$\text{Leakage energy} = 200{,}000 \int_0^{t_0} N dt = 100{,}000\, N_0 t_0.$$

Since the acceleration is supposed to be uniform, the energy dissipated by the brakes is

$$pA_1fr \int_0^{t_0} N dt = \frac{pA_1frN_0}{2} t_0,$$

where p = pressure at brake surface in pounds per square inch,
A_1 = total area covered by brake shoes,
A_2 = total area of brake plate,
f = coefficient of friction,
r = mean radius in feet,
N_0 = initial speed of rotor in radians per second,
t_0 = time in seconds required to stop the machine.

Equating energies, we have

$$\frac{pA_1frN_0t_0}{2} = 4.48 \times 10^8 + 100{,}000\, N_0t_0.$$

The table of constants is:

$$A_1 = 20(9 \times 18) = 3240 \text{ in.}^2$$
$$A_2 = 2\pi \times 171 \times 10(2.54)^2 = 69{,}300 \text{ cm.}^2$$
$$N_0 = 88.3 \text{ r.p.m.} = 9.24 \text{ radians per sec.}$$
$$pA_1fr = 3240 \times 0.25 \times \frac{171}{12} p = 11{,}540\, p \text{ lb. ft.},$$

where p is in pounds per square inch. Thus, from the equation of energies above,

$$t_0 = \frac{8.41 \times 10^3}{p - 17.33}.$$

Substituting this value of t_0 in (343), we have

$$\text{Max. } e_0 = \frac{2}{3} i_0 \left[\frac{1.682R \times 10^4}{\pi C(p - 17.33)}\right]^{\frac{1}{2}}.$$

The constants of this equation are:

$$\frac{1}{R} = 0.46 \frac{\text{watt cm.}}{\text{cm.}^2\ {}^\circ\text{C.}} (69{,}300 \text{ cm.}^2) = 31{,}800 \frac{\text{watt cm.}}{{}^\circ\text{C.}}$$

$$C = 3.9 \frac{\text{watt sec.}}{\text{cm.}^3\ {}^\circ\text{C.}} (69{,}300 \text{ cm.}^2) = 270{,}000 \frac{\text{watt sec.}}{\text{cm.}}$$

$$i_0 = 11{,}540\, p \text{ ft-lb.} \times 9.24 \frac{\text{radians}}{\text{sec.}} = 106{,}600\, p \frac{\text{ft-lb.}}{\text{sec.}}$$
$$= 144{,}700\, p \text{ watts.}$$

Finally then,

$$\text{Max. } e_0 = \frac{76.0\, p}{\sqrt{p - 17.33}}.$$

This is the result required.

Maximum e_0 is a minimum if $p = 34.7$ lb. per sq. in., and for this pressure

$$\text{Max. } e_0 = 634\ {}^\circ\text{C.}$$

and

$$t_0 487 \text{ sec.} = 8.12 \text{ min.}$$

For a brake pressure of 100 lb. per sq. in.

$$\text{Max. } e_0 = 836\ {}^\circ\text{C.}$$

and

$$t_0 = 1.69 \text{ min.}$$

129. Switching. Mathematically, those phases of the operational calculus so far studied deal with the application of the unit e.m.f. at $t = 0$. Physically, this is accomplished by closing a switch in that mesh of the circuit which contains a battery. Since the operational calculus handles such a case so effectively, we are led to inquire whether it will also handle cases where switches are opened or where switches not in the battery mesh are closed.

Consider the circuit shown in Fig. 82. At $t = 0$, the circuit was energized by the insertion of a unit e.m.f. at B. By means of the methods developed above, the voltage appearing at the terminals A may be calculated as a function of time. It is apparent that a generator supplying this particular voltage may be connected across A without affecting the currents in any part of the circuit. If, at $t = t_1$, a voltage, equal in magnitude but opposite in sign, be added in series with this generator, the net effect

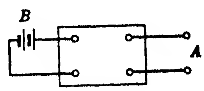


FIG. 82.

will be to produce zero voltage across A. This is exactly the result of closing a switch at A at time $t = t_1$. Evidently the resulting current in any mesh may be found by superposing the current due to the reversed generator at A upon that which would have existed had A been left open.

EXAMPLE 1. An inductance L is in series with a resistance R. At time $t = 0$, a steady voltage E is inserted into the circuit. At time $t = t_1$, the resistance is short-circuited. Find the current at any instant.

The impedance operator for the original circuit is $Z(p) = R + Lp$, whence the original current is

$$i_1 = \frac{E}{R + Lp} 1 = \frac{E}{R}(1 - e^{-at}),$$

where $a = R/L$. The voltage across the resistance is

$$e_1 = i_1 R = E(1 - e^{-at}).$$

If time be measured from $t = t_1$ instead of $t = 0$, this voltage would become

$$e_2 = E[1 - e^{-a(t+t_1)}].$$

If a generator producing a voltage $- e_2$ be placed in the circuit instead of R, the current will be unchanged. The effect of short-circuiting R is then to introduce a generator giving $+ e_2$ volts. The resistance of the battery being negligible, the impedance operator for e_2 is $Z(p) = Lp$. That is,

$$i_2 = \frac{1}{Lp} e_2 1.$$

This may be evaluated by means of the superposition theorem, i.e.,

$$A(t) = \frac{1}{Lp} 1 = t/L,$$

$$e(t) = e_2 = E(1 - e^{-a(t+t_1)}).$$

Then

$$\begin{aligned} i_2 &= e(0)A(t) + \int_0^t A(t - \lambda)e'(\lambda)d\lambda \\ &= \frac{Et}{L}(1 - e^{-at_1}) + \int_0^t \frac{t - \lambda}{L} \cdot Eae^{-a(\lambda+t_1)}d\lambda \\ &= \frac{Et}{L} - \frac{E}{R}e^{-at_1}(1 - e^{-at}). \end{aligned}$$

The zero of time must now be returned to $t = 0$. This is accomplished by writing $t - t_1$ for t. Then

$$i'_2 = \frac{E(t - t_1)}{L} - \frac{E}{R} e^{-at_1}[1 - e^{-a(t-t_1)}].$$

This is the increment of current due to short-circuiting the resistance. The total current is therefore

$$i = i_1 + i'_2 = \frac{E}{L}(t - t_1) + \frac{E}{R}(1 - e^{-at_1}),$$

when $t > t_1$.

The current increases linearly with time, as would be expected.

The problem of opening a switch at some point in a circuit is not fundamentally different from that of closing a switch. In this case, however, a current generator rather than a voltage generator must be inserted.

Example 2. Suppose that in the circuit of the previous example the resistance is initially in the circuit but short-circuited, and at $t = t_1$ this short circuit is removed. We desire the current at any instant. The initial current is

$$i_0 = \frac{E}{Lp} 1 = Et/L,$$

or referred to the time t_1,

$$i'_0 = E(t + t_1)/L.$$

The current i_0 flows through the switch until $t = t_1$, at which time the switch is opened. This is equivalent to inserting a generator producing a current i'_0 at $t = 0$. This generator acts on a circuit consisting of R and L in parallel. The impedance operator for this circuit is

$$Z(p) = \frac{RLp}{R + Lp} = \frac{Rp}{a + p}.$$

The voltage across the parallel circuit is

$$e = i'_0 z$$

whence the current through the resistance is

$$i_R = E/R = i'_0 z/R.$$

This expression may be evaluated by the superposition theorem. Although this theorem was derived for current in terms of indicial

admittance, it can also be applied to give the current in terms of the impedance function as in this case. Here

$$A(t) = Z(p)\,1 = \frac{Rp}{a+p}\,1 = Re^{-at}$$

$$e(\lambda) = i'_0(\lambda) = E(\lambda + t_1)/L$$

whence

$$i_R = e(0)A(t) + \int_0^t A(t-\lambda)e'(\lambda)d\lambda$$

$$= Eat_1e^{-at} + \int_0^t Eae^{-at}e^{a\lambda}d\lambda$$

$$= E[1 + (at_1 - 1)e^{-at}].$$

Shifting the time axis back to $t = 0$,

$$i'_R = E[1 + (at_1 - 1)e^{-a(t-t_1)}].$$

This is the increment of current through R due to opening the switch. It is to be added to the current originally present in R; since R was originally short-circuited, the latter current is zero, and i'_R is the actual current in the circuit after $t = t_1$.

EXERCISES AND PROBLEMS

Prove the two shifting formulas:

1. $$\frac{1}{Z(p)}\,e^{-bt}\,1 = e^{-bt}\,\frac{1}{Z(p-b)}\,1.$$

2. $$\frac{1}{Z(p)}\,1 = e^{-bt}\,\frac{p}{p-b}\,\frac{1}{Z(p-b)}\,1.$$

3. In transoceanic cables, it is customary to "load" the line by either inserting inductance coils at frequent intervals along the length or by wrapping the wire with a magnetic tape. In this case, the leakage is small, but L, R, and C must all be considered. The entering current then is

$$i_0 = \sqrt{\frac{C}{L}}\left(1 + \frac{a}{p}\right)^{-\frac{1}{2}} 1, \text{ where } a = \frac{R}{L}.$$

Find i_0 as a function of the time.

4. In the circuit of the Fig. 82a, the switch S is closed at time $t = 0$ and S_1 is closed at time t_0. Find the current at time t subsequent to t_0.

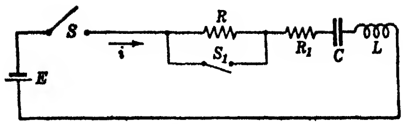


FIG. 82a.

5. Solve the example of § 128 subject to the assumptions:

(a) That the thickness of the brake plate has an effect on the surface temperature.

(b) The loss of heat from the sides of the brake plate is negligible.

Consider the problem to be equivalent to an open-circuited transmission line whose length is b. (See example 2, § 125.)

6. From the results of problem 5 plot a curve of temperature in degrees Centigrade of the brake plate surface against time, after applying the brakes, up to the time when the machine stops. Use the value of the thickness b and the hydraulic cylinder pressure given below.

b—inches	Pressure on Brake Plate Surface, lb. per sq. in.	b—inches	Pressure on Brake Plate Surface, lb. per sq. in.
2.00	55	2 00	200
2.00	60	0.50	100
2.00	70	0.75	100
2.00	80	1.00	100
2.00	90	1.25	100
2.00	100	1.50	100
2.00	110	1.75	100
2.00	120	2.25	100
2.00	130	2.50	100
2.00	140	3.00	100
2.00	150	3.50	100
2.00	160	4.00	100
2.00	170	5.00	100
2.00	180	6.00	100
2.00	190		

The operational calculus is not applicable in the analysis of non-linear circuits. For a new theory of non-linear circuits containing variable inductance, variable capacitance, and variable resistance see Ref. 6 at the end of this volume.

REFERENCES TO ADDITIONAL AND RELATED MATERIAL OF THE TEXT

I. Differential Equations

Topic	Publication
1. Exact Differential Equations and Integrating Factors.	Murray, *Introductory Course in Differential Equations*, p. 17. Kells, *Elementary Differential Equations*, p. 29.
2. Homogeneous Linear Equation. $(t^n p^n + a_1 t^{n-1} p^{n-1} + \ldots + a_n)y = f(t)$.	Murray, *loc. cit.*, p. 82. Kells, *loc. cit.*, p. 117.
3. Transformation of Second-order Equation into Integrable Form by Change of Dependent and Independent Variable.	Murray, *loc. cit.*, p. 114.
4. Integration in Series of Differential Equations.	Murray, *loc. cit.*, p. 101. Kells, *loc. cit.*, p. 120.
5. Graphical and Numerical Integration of Differential Equations.	See Sec. VI of Chap. II.
6. Integration of Analytic Non-linear Differential Equations.	Keller, *Mathematics of Modern Engineering*, Vol. II.
7. Set-up of Differential Equations of Motion of Complicated Mechanical and Electrical Systems by Lagrange's Equations and Tensors.	Keller, *loc. cit.*
8. Derivation of the Differential Equations of Vibrating Systems.	Kimball, *Vibration Prevention in Engineering* (General Electric Co. Series).
9. Linear Transmission Line Circuits.	Johnson, *Transmission Circuits of Telephone Communication* (Bell Lab. Series).
10. Analogies between Mechanical and Electrical System.	Nickle, "Oscillographic Solution of Electromechanical Systems," *Trans. A.I.E.E.*, 54 (1925), p. 847.

II. Determinants

Topic	Publication
11. Proof of Laplace's Expansion.	Dickson, *Elementary Theory of Equations*, p. 141.
12. Factoring Determinants.	Scott and Matthews, *Theory of Determinants*, p. 75.
13. Derivative of Determinants.	Scott and Matthews, *loc. cit.*, p. 39.
14. Symmetric, Skew, and Skew-Symmetric Determinants.	Scott and Matthews, *loc. cit.*, p. 88. Muir, *Theory of Determinants*, Vol. I, p. 395. Burnside and Panton, *Theory of Equations*, Vol. II, p. 43.
15. Inverse or Reciprocal Determinants.	Burnside and Panton, *loc. cit.*, p. 41.

Topic	Publication
16. Infinite Determinants.	Scott and Matthews, *loc. cit.*, p. 120. Whittaker and Watson, *A Course of Modern Analysis*, p. 36.
17. Systems of Fewer than n Linear Equations and Systems of More than n Linear Equations in n Unknowns.	Dickson, *Elementary Theory of Equations*, p. 144. Dickson, *Modern Algebraic Theories*, p. 60.
18. Matrices.	Dickson, *loc. cit.*, p. 39.

III. Fourier Series

19. Use of Fourier Series in Flywheel Design.	Doherty and Franklin, "Design of Flywheels for Reciprocating Machinery Connected to Synchronous Generators or Motors," *Trans. A.S.M.E.*, 42 (1920), p. 523.
20. Proof of the Convergence of Fourier Series to the Function $f(x)$.	Carslaw, *Fourier Series and Integrals*, p. 230.
21. General Theorems on Convergence, Integration, and Differentiation of Fourier Series.	Carslaw, *loc. cit.*, Chap. VIII.
22. Additional Methods of Fourier Analysis.	Carse and Shearer, *A Course in Fourier's Analysis and Periodogram Analysis* (Edinburgh Math. Tracts).
23. Treatise on Theory of Fourier Series.	Hobson, *The Theory of Functions of a Real Variable and the Theory of Fourier's Series*, p. 477.

IV. Solution of Higher-degree and Transcendental Equations

24. Algebraic Formulas for the Solution of Cubic and Quartic Equations.	Burnside and Panton, *loc. cit.*, Vol. I, p. 115. Dickson, *Elementary Theory of Equations*, p. 31.
25. Graeffe's Method.	Whittaker and Robinson, *Calculus of Observations*. Scarborough, *Numerical Mathematical Analysis*, p. 198.

V. Dimensional Analysis

26. General Reference on Dimensional Analysis.	Bridgman, *Dimensional Analysis*.
27. Original Proof of Formulas for Change of Units.	Buchholz, *Ann. Phys.*, 51 (1916), p. 678.
28. Buckingham's Proof of the Theorem.	Buckingham, *Phys. Rev.*, 4 (1914), p. 345.
29. Tolman's Principle of Similitude.	Tolman, *Phys. Rev.*, 3 (1914), p. 244.
30. Heat Transfer in Pipes.	Nusselt, *Zeitschrift der Verein Deutscher Ingenieur*, 53 (1909), pp. 1750–1808.

VI. Numerical Method of Solving Differential Equations

Topic	Publication
31. Numerical Integration of Systems of Differential Equations.	Moulton, *New Methods in Exterior Ballistics*, p. 60.
	Scarborough, *Numerical Mathematical Analysis*, p. 218.
	Adams, *Smithsonian Mathematical Formulae and Tables of Elliptic Functions*, Chapter X.
	Whittaker and Robinson, *The Calculus of Observations*.
32. Differential Analyzer. A New Machine for Solving Differential Equations.	Bush, *Journal of the Franklin Institute*, (1931), p. 447.

VII. Vector Analysis

Topic	Publication
33. Proofs of Green's, Stokes', and Gauss' Theorems.	Coffin, *Vector Analysis*, pp. 112, 124, 148.
	Mason and Weaver, *The Electromagnetic Field*, pp. 102, 348, 349.
	Gibbs, *Vector Analysis*, pp. 184, 187, 197.
34. Scalar Potential.	Kellogg, *Foundation of Potential Theory*
35. Vector Potential.	Stevenson and Park, *General Electric Review*, 31 (1928), pp. 89, 153.
	Mason and Weaver, *The Electromagnetic Field*, p. 191.
	Hague, *Electromagnetic Problems in Electrical Engineering*, p. 77.
36. Behavior of Vector Potential and Flux Density at Boundaries.	Stevenson, *General Electric Review*, 29 (1926), p. 797.
	Mason and Weaver, *The Electromagnetic Field*, pp. 197 and 212.
	Hague, *Electromagnetic Problems in Electrical Engineering*, Chap. IV.
37. Derivation of the Partial Differential Equations of Mathematical Phys-	Webster and Szego, *Partielle Differentialgleichungen der mathematischen Physik*, Chap. I.
	Weatherburn, *Advanced Vector Analysis*, Vol. 2.
38. Dyadics.	Gibbs, *Vector Analysis*, Chap. V.
	Weatherburn, *Advanced Vector Analysis*, Chaps. V, VI, VII, and VIII.

VIII. Operational Calculus

Topic	Publication
39. Bromwich's Original Proof of Heaviside's Results. (Normal Coordinates in Dynamical Systems.)	Bromwich, *London Math. Soc.*, 15 (1916), p. 401.
40. K. W. Wagner's Proof of Heaviside's Results.	Wagner, *Archiv für Elek.*, 4 (1916), p. 159.

Topic	Publication
41. Proofs of Operational Calculus Theory by Means of Integral Equations.	Carson, *Electric Circuit Theory and Operational Calculus.*
42. Proof of the Equivalence of the Methods of Bromwich and Carson.	March, *Bull. Am. Math. Soc.*, 33 (1927), p. 311.
43. Bromwich's Treatment of Fractional Exponents.	Bromwich, *Proc. Camb. Phil. Soc.*, 20 (1921), p. 411.
44. Operational Methods in the Solution of Partial Differential Equations.	Jeffreys, *Operational Methods in Mathematical Physics.*
45. Impulsive Networks.	Bush, *Operational Circuit Analysis.*
46. Extensive Treatment of Transmission Line Transients.	Berg, *Heaviside's Operational Calculus.*
47. Linear Heat Flow; Switching Operations; and Bibliography on Operational Methods.	Berg, *Heaviside's Operational Calculus.*

ANSWERS

Pages 21-22

1. $mx'' - kx = 0.$

2. $l\theta'' + g \sin \theta = 0.$

3. $mr^2\theta'' + 2k\theta = 0.$

4. $ms'' + 62.4\pi r^2 s = 0$, where s is the displacement below equilibrium position at time t.

5. $M_2 s''_2 + k(s'_2 - s'_1) + k_2(s_2 - s_1) = 0.$
   $M_1 s''_1 - k(s'_2 - s'_1) + k_1 s_1 - k_2(s_2 - s_1) = 0.$

6. $a(m_1 + m_2)\theta''_1 + bm_2\theta''_2 + (m_1 + m_2)g\theta_1 = 0,$
   $a\theta''_1 + b\theta''_2 + g\theta_2 = 0$, where g is the acceleration of gravity.

Page 34

1. $y = C_1 e^t + C_2 e^{-t} + C_3 e^{7t}.$

2. $y = C_1 e^{-t} + e^{-t/2}(C_2 \sin \sqrt{\tfrac{3}{2}}\, t + C_3 \cos \sqrt{\tfrac{3}{2}}\, t).$

3. $y = C_1 e^{\alpha t} + C_2 t e^{\alpha t}.$

4. $y = e^{\alpha t}(C_1 \sin kt + C_2 \cos kt).$

5. $y = C_1 e^{2t} + C_2 e^{-2t} - \frac{1}{29} \sin 5t.$

6. $y = C_1 e^t + C_2 e^{2t} + \frac{1}{2} e^{3t}.$

7. $y = C_1 e^{kt} + C_2 e^{2t} + \dfrac{t}{k-2} e^{kt}, \quad (k \neq 2).$

8. $y = C_1 \sin kt + C_2 \cos kt + t(C_3 \sin kt + C_4 \cos kt) - \dfrac{1}{8k^2} t^2 \sin kt.$

9. $\theta = \theta_0 \cos \left(\dfrac{2k}{mr^2}\right)^{\frac{1}{2}} t.$

10. $s = -l \cos \left(\dfrac{62.4\pi r^2}{m}\right)^{\frac{1}{2}} t$, Period $= \left(\dfrac{2m\pi}{31.2r^2}\right)^{\frac{1}{2}}$, Amplitude $= l$.

11. $\theta = \phi \left(\dfrac{l}{g}\right)^{\frac{1}{2}} \sin \sqrt{\dfrac{g}{l}}\, t + \theta_0 \cos \sqrt{\dfrac{g}{l}}\, t.$

Pages 42-44

1. $y = (C_1 + C_2 t)e^{\alpha t}.$

2. $y = C_1 \sin t + C_2 \cos t + \dfrac{t}{5} e^{2t} - \dfrac{4}{25} e^{2t}.$

3. $x = C_1 \sin at + C_2 \cos at + \dfrac{2D}{(a^2 - \omega^2)} \cos \omega t + \dfrac{D \cos 2\omega t}{2(a^2 - 4\omega^2)} + \dfrac{E}{a^2}.$

4. $y = (C_1 e^t + C_2 e^{-t}) \cos t + (C_3 e^t + C_4 e^{-t}) \sin t + \frac{1}{85} (\sin 3t + \cos 3t).$

5. $y = e^{-\alpha t}(C_1 \sin \beta t + C_2 \cos \beta t) + \sum_{n=1}^{n=3} (D_n a_n \sin nt + E_n b_n \cos nt),$

where $\alpha = \dfrac{R}{2L}$, $\beta = \dfrac{1}{2CL}(4CL - C^2R^2)^{1/2}$, $c = \dfrac{R}{L}$, $d = \dfrac{1}{CL}$, $f = \dfrac{b_n}{a_n}$,

$$D_n = \frac{(d - n^2) + ncf}{(d - n^2)^2 + n^2c^2}, \quad E_n = \frac{(d - n^2) - ncf^{-1}}{(d - n^2)^2 + n^2c^2}.$$

8. $y = t^2(C_1 t^{\sqrt{3}i} + C_2 t^{-\sqrt{3}i}) + \frac{1}{4}t^2.$

9. (a) $\tan y = C(1 - e^x)^3.$

(b) $y = 1 - (\sqrt{1 - x} - C)^2.$

(c) $\cos y = C \cos x.$

10. (a) $y = \dfrac{1}{x}(C_1 + \log x).$

(b) $y = C_1 e^{-x^2/2} - 2 + x^2.$

(c) $y = C_1 e^{-\sin \theta} - 2 + 2 \sin \theta.$

11. (a) $y = \dfrac{1}{t}(C_1 + C_2 \log t).$

(b) $y = C_1 t^2 + C_2 t^{-3} + \frac{3}{5}t^2 \log t - \frac{2}{3}.$

(c) $y = \dfrac{1}{t}(C_1 + C_2 \log t) + \dfrac{t}{4} \quad \dfrac{t^3}{16 \times 3!} + \quad 36 \times 5!$

Pages 52-54

1. $s_1 = C_1 e^{m_1 t} + C_2 e^{m_2 t} + C_3 e^{m_3 t} + C_4 e^{m_4 t},$

$s_2 = C_1 a_1 e^{m_1 t} + C_2 a_2 e^{m_2 t} + C_3 a_3 e^{m_3 t} + C_4 a_4 e^{m_4 t},$

where $a_i = \dfrac{k_d m_i + k_2}{M_2 m_i^2 + k_d m_i + k_2}$ and m_i, $(i = 1, 2, 3, 4)$ are the roots of the characteristic equation

$$\begin{matrix} -(k_d p + k_2) & M_2 p^2 + k_d p + k_2 \\ M_1 p^2 + k_d p + k_1 + k_2 & -(k_d p + k_2) \end{matrix}$$

2. $\theta_1 = C_1 e^{i\omega_1 t} + C_2 e^{i\omega_2 t} + C_3 e^{i\omega_3 t} + C_4 e^{i\omega_4 t},$

$\theta_2 = C_1 a_1 e^{i\omega_1 t} + C_2 a_2 e^{i\omega_2 t} + C_3 a_3 e^{i\omega_3 t} + C_4 a_4 e^{i\omega_4 t},$

where $\omega_1 = -\omega_3 = \sqrt{B + T}$, $\omega_2 = -\omega_4 = \sqrt{B - T}$,

$B = \dfrac{(m_1 + m_2)(a + b)g}{2m_1 ab}$ $\quad T = \dfrac{g\{[(m_1 + m_2)(a + b)]^2 - 4m_1 ab(m_1 + m_2)\}^{1/2}}{2m_1 ab}$, and

$g - b\omega_i^2$

3. $q_1 = B_0 + \sum_{i=1}^{2} B_i e^{m_i t}$, $q_2 = B'_0 + \sum_{i=1} B'_i e^{m_i t}$, $i_1 = \sum_{i=1} m_i B_i e^{m_i t}$,

where $B'_i = \dfrac{L_{12}m_i + R_{12}}{L_{12}m_i + R_2 + R_{12}} B_i = \alpha_i B_i,$

0, m_1, m_2 are the roots of the characteristic equation and, due to the initial conditions, B'_0, B_1, B_2, satisfy the linear equations

$$\sum_{i=1}^{2} B_i = -C_1E, \quad \sum_{i=1}^{2} m_iB_i = 0, \quad \sum_{i=1}^{2} \alpha_iB_i = -B'_0.$$

4. The general solution of the differential equations is:

$$q_1 = e^{\alpha_1 t}(A_1 \sin \omega_1 t + A_2 \cos \omega_1 t) + e^{\alpha_2 t}(A_3 \sin \omega_2 t + A_4 \cos \omega_2 t) + q_{1s},$$

$$q_2 = e^{\alpha_1 t}(A'_1 \sin \omega_1 t + A'_2 \cos \omega_1 t) + e^{\alpha_2 t}(A'_3 \sin \omega_2 t + A'_4 \cos \omega_2 t) + q_{2s},$$

where $\Delta(j\omega)$ is the characteristic determinant with p replaced by $j\omega$, and

$$q_{1s} = \frac{B_2E}{C_2C_{12}|\Delta(j\omega)|} \sin (\omega t + \phi_1 - \phi_2)$$

$$q_{2s} = \frac{(1 - C_{12}L_{12}\omega^2)E}{C_{12}|\Delta(j\omega)|} \sin (\omega t - \phi_2),$$

$$B_2 \quad \{C_2^2C_{12}^2(L_2 + L_{12})^2\omega^4 + [R_2^2C_{12}^2C_2^2 - 2C_2C_{12}(C_2 + C_{12})(L_2 + L_{12})]\omega^2 + (C_2 + C_{12})^2\}^{1/2},$$

$$\phi_2 = \text{arc cot} \frac{\text{real part of characteristic determinant}}{\text{coefficient of imaginary part of characteristic determinant}},$$

$$\phi_1 = \text{arc tan} \frac{R_2C_{12}C_2\omega}{C_2 + C_{12} - C_{12}C_2(L_1 + L_{12})\omega^2},$$

$\alpha_1 \pm j\omega_1$, $\alpha_2 \pm j\omega_2$ are the roots of the characteristic equation and the A'_i are related to A_i by means of the differential equation for the second branch of the circuit. (For methods of obtaining the above roots see §§ 41, 46.)

5. $$L_1 \frac{d^2i_1}{dt^2} + L_{12}\frac{d^2i_2}{dt^2} + R_1\frac{di_1}{dt} + \frac{1}{C_1}i_1 = E\omega \cos \omega t,$$

$$L_{21} \frac{d^2i_1}{dt^2} + L_2\frac{d^2i_2}{dt^2} + \frac{1}{C_2}i_2 - \frac{1}{C_2}i_3 = 0$$

$$R_3\frac{di_3}{dt} - \frac{1}{C_2}i_2 + \frac{1}{C_2}i_3 \quad 0.$$

7. $x'' + \dfrac{gkl^2x}{k_1^2W} = 0$, where x is the vertical displacement of the upper end of the spring and k radius of gyration about B.

$$\frac{2\pi k_1}{l}\sqrt{\frac{W}{gk}}.$$

8. $z'' + \dfrac{4\lambda}{M}z = 0, \quad \theta'' + \dfrac{4l^2\lambda\theta}{k^2M} \quad 0, \quad T_z = \pi\sqrt{\dfrac{M}{\lambda}}, \quad T_\theta = \dfrac{\pi k}{l}\sqrt{\dfrac{M}{\lambda}}$

where $2l$ is the length of the beam, k is the radius of gyration about the center of gravity of the beam, and λ is the spring constant of one spring.

9. $\theta'' = \dfrac{2g}{3r}$

10. $\theta'' + \frac{g}{r}(f \sin \theta - \cos \theta) = 0$, where f is the coefficient of friction.

11. $\theta'' + 27.6\theta = 0$, $\theta = 0.5833 \cos 5.25t$, period $= 1.196$ seconds.

12. $v = 32.2t - 13.33$ = velocity of car at any time after cable breaks until car strikes air-cushion.

$\frac{dx}{dt} = v = [632.6x - 6218(100 - x)^{-.4} + 40{,}663]^{1/2}$ = velocity of car after striking air-cushion, where x is distance measured downward from the point where air-cushion begins to act.

Page 72

1. (*a*) $x = -8$, $y = -7$, $z = 26$. (*b*) $x = 1$, $y = 2$, $z = 3$, $w = 5$.
   (*c*) No solution. (*d*) $x = z = 0$, $y = 2w'$.
   (*e*) $x = \frac{11 + 2w'}{21}$, $y = \frac{32 + 2w'}{21}$, $z = \frac{213 - 30w'}{21}$.
   (*f*) $x = y = z = w = 0$.

$$\begin{vmatrix} m^2L_1 + m(R_1 + R_{12}) + \frac{1}{C_1} & -mR_{12} \\ -mR_{12} & m^2L_2 + m(R_2 + R_{12}) + \frac{1}{C_2} \end{vmatrix} = 0.$$

$$\begin{vmatrix} m^2L_1 + mR_1 + \frac{1}{C_1} + \frac{1}{C_{12}} & -\frac{1}{C_{12}} \\ -\frac{1}{C_{12}} & m^2L_2 + mR_2 + \frac{1}{C_{12}} \end{vmatrix} = 0.$$

$$\begin{vmatrix} m^2L_1 + mR_1 + \frac{1}{C_1} & m^2L_{12} \\ m^2L_{12} & m^2L_2 + mR_2 + \frac{1}{C_2} \end{vmatrix} = 0.$$

3. $q_1 = \left|\frac{z_{22}}{\Delta(j\omega)}\right| E \sin(\omega t - \phi_1)$,

$q_2 = \left|\frac{z_{21}}{\Delta(j\omega)}\right| E \sin(\omega t - \phi_2)$,

where $\Delta(j\omega) = z_{11}(j\omega)z_{22}(j\omega) - z_{21}^2(j\omega)$,

$z_{11} = \frac{1}{C_1} + (R_1 + R_{12})j\omega - L_1\omega^2$,

$z_{22} = \frac{1}{C_2} + (R_1 + R_{12})j\omega - L_2\omega^2$,

$z_{12} = z_{21} = -R_{12}j\omega$.

Pages 128-130

1. (*a*) 1.357, 1.692, −3.049.
   (*b*) 1.576, 0.491, −2.067.
   (*c*) 1.247, −1.802, −0.445.

(*d*) Six roots are: ± 1.875, ± 4.694, ± 7.855.

(*e*) -1.934.

(*f*) 0.607.

2. (*a*) -1, 3, 0.2361, -4.236.

(*b*) 1, 3, 5, 7.

(*c*) 10, -3.40, 3, 2.1.

(*d*) -3.450, 1.450, $0.5 \pm 1.32i$.

3. -4.861, 4.000, 4.000, -3.254, -0.8851.
4. 0.396, 1.349, -3.745, $1.461 \pm i1.641$, $-0.4607 \pm i1.285$.
5. -0.3002, -0.1602, -0.09888, -0.0579, -0.002084.
6. Yes. Moreover, the roots are $-4.35 \pm 20.2i$, $-8.39 \pm 6.36i$.

Pages 205-206

1. 24.5.
2. -27.6, $40i - 18j - 16k$.

   34.7, $-60i + 3.9j - 6.9k$.

5. $(ix + jy + kz) \cdot \nabla\phi = -\dfrac{1}{(x^2 + y^2 + z^2)^{1/2}}$.

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3.
$$\begin{aligned} A = -j \Big\{ &(a - x)(a - y) \log [(a - x)^2 + (a - y)^2] \\ &+ (a + x)(a - y) \log [(a + x)^2 + (a - y)^2] \\ &+ (a - x)(a + y) \log [(a - x)^2 + (a + y)^2] \\ &+ (a + x)(a + y) \log [(a + x)^2 + (a + y)^2] \\ &+ (a - x)^2 \left[\arctan \frac{a - y}{a - x} + \arctan \frac{a + y}{a - x}\right] \\ &+ (a + x)^2 \left[\arctan \frac{a - y}{a + x} + \arctan \frac{a + y}{a + x}\right] \\ &+ (a - y)^2 \left[\arctan \frac{a - x}{a - y} + \arctan \frac{a + x}{a - y}\right] \\ &+ (a + y)^2 \left[\arctan \frac{a - x}{a + y} + \arctan \frac{a + x}{a + y}\right]\Big\}, \end{aligned}$$

where a side of the square is $2a$ and $P(x, y)$ is any point exterior to the square

Page 239

1. $Y = gI + b(I \times k)$.

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2. $\dfrac{5}{12}\pi$.

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12. $i_1 = 10.0 - 9.55e^{-11.88t} - 0.45e^{-2.42t}$,

$i_2 = 3.03(e^{-2.42t} - e^{-11.88t})$.

Pages 298-299

3. $i = \sqrt{\dfrac{C}{L}}\, e^{-(a/2)t}\left[1 + \dfrac{(at/4)^2}{(1!)^2} + \dfrac{(at/4)^4}{(2!)^2} + \dots\right] = \sqrt{\dfrac{C}{L}}\, e^{-(a/2)t} J_0(ait/2)$,

where J_0 is the Bessel function of order zero and $i = \sqrt{-1}$.

4. $i = \dfrac{e^{-bt}}{L\omega} \sin \omega t + \dfrac{R}{L^2\omega^2} e^{-(b-b_1)t_0 - b_1t} \sin \omega t_0 \sin \omega(t - t_0)$

$$+ \frac{R}{L^2\omega^2} \int_0^{t-t_0} e^{-b_1t - (b-b_1)t_0 - (b-b_1)\lambda} \sin \omega(t - t_0 - \lambda)\,[\omega \cos \omega(\lambda + t_0)$$

$$- b \sin \omega(\lambda + t_0)]d\lambda,$$

where $b = \dfrac{R + R_1}{2L}$, $b_1 = \dfrac{R_1}{2L}$, $\omega = \dfrac{1}{(LC)^{1/2}}$, $E =$ unity.

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